ON THE COMMUTATIVE FAMILY OF SUBNORMAL OPERATORS

By

Takasi ITÔ

Introduction. HALMOS has given in [3] the definition of a subnormal operator and the characteristic property of it. A bounded operator A defined on a HILBERT space $\mathfrak{H}$ is said to be *subnormal* if there exist a HILBERT space $\mathfrak{K}$ containing $\mathfrak{H}$ and a bounded normal operator N on $\mathfrak{K}$ such that $Ax=Nx$ for every x in $\mathfrak{H}$. Recently in [1] BRAM has made HALMOS' characterization simpler ([1], Theorem 1) and given another characteristic property ([1], Theorem 2) and some results about subnormal operators (for example, [1], Theorems 4, 7, 8, 9).

In this paper first we shall study the problem under what conditions it is possible to extend the commutative family of subnormal operators acting on a HILBERT space $\mathfrak{H}$ to the commutative family of normal operators on a HILBERT space $\mathfrak{K}$ containing $\mathfrak{H}$. Theorem 1 answers to this question. Then we shall give a generalization of BRAM's theorems (for example Theorem 6 and Theorem 7) and another simpler proof of BRAM's theorm about the spectrum of subnormal operators (Theorem 8). Theorem 3 is a generalization of COOPER's result in [2] (cf. [9], p. 393). Theorem 5 gives a new characterization of subnormal operators.

1. An abelian semi-group of subnormal operators. Throughout the paper, a HILBERT space is a vector space over the complex numbers, an operator is a bounded linear transformation unless denoted explicitly. For an operator A we denote by A^* an adjoint operator of A.

Lemma 1. *Let $A_l\,(l=1,2,\cdots,n)$ be n commutative operators on a HILBERT space $\mathfrak{H}$. If for every non-negative integer M and element $x_{i_1,i_2,\cdots,i_n}$ in $\mathfrak{H}$ $(0\leqq i_l\leqq M,\ l=1,2,\cdots,n)$*

$$(1.1)\qquad \sum_{\substack{i_l,j_l\geqq 0\\ l=1,2,\cdots,n}}^{M} (A_1^{i_1}A_2^{i_2}\cdots A_n^{i_n}x_{j_1,j_2,\cdots,j_n},\ A_1^{j_1}A_2^{j_2}\cdots A_n^{j_n}x_{i_1,i_2,\cdots,i_n})\geqq 0\ ,$$

then we have the inequality such that for every M, $x_{i_1,i_2,\cdots i_n}$ in $\mathfrak{H}$ $(0\leqq i_l\leqq M,\ l=1,2,\cdots,n)$ and non-negative integer $\nu_l\ (l=1,2,\cdots,n)$