ON THE CLASS OF FUNCTIONS INTEGRABLE IN A CERTAIN GENERALIZED SENSE

Dedicated to Professor Kinjiro Kunugi on the occasion of his sixtieth birthday

By

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1. Introduction. Let us consider the Lebesgue integral on a bounded interval I of the real line. There have been many attempts of extending the concept of integral for a larger class of measurable functions; some of them depend heavily on special properties of the derivative on the real line, preventing to be brought in general settings, but some are based only on the notions of measure theory, admitting investigation in general cases. We will follow the latter direction.

If a measurable function is not integrable, it has points of singularity, i.e., roughly speaking, it is not integrable on any neighborhood containing the points. If functions with a fixed point of singularity are in question, a generalized integral may be defined as something like Cauchy's principal value. However, if points of singularity are distributed over some region, depending on a function $f(x)$, such an approach must undergo some modification. A natural generalization will be

$$\lim_{n\to\infty}\int_{F_n} f(x)\,dx\,,$$

where $\{F_n\}$ is a sequence of measurable sets such that

$$F_n\subset F_{n+1}\qquad (n=1,2,\cdots)\qquad\text{and}\qquad \lim_{n\to\infty}\operatorname{meas}(F_n^c)=0$$

with F_n^c denoting the complement of F_n. Since the region of singularity varies along the function $f(x)$, we can not take one and the same sequence $\{F_n\}$ for the definition of a generalized integral, and even for a single function the limit on a sequence $\{F_n\}$ may differ from that on another $\{F_n'\}$. In order to avoid these inconvenience and ambiguity, sequences used for the definition must obey some additional requirements. In this respect, Kunugi [3] proposed a definition

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