SOME STUDIES ON HOMOLOGICAL ALGEBRA

By

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§ 1. Let A be an algebra over a commutative ring K, and A^e the enveloping algebra of A : $A^e = A \otimes_K A^*$, A^* being the opposite algebra of A. In this paper, we shall mostly assume that A satisfies the condition that A^e is projective as a left A^*-module (or equivalently, as a right A-module), which was first considered by Azumaya [1]. The class of such algebras contains that of algebras which are projective as K-modules. Cartan and Eilenberg proved in [2] that the cohomology groups $H^n(A, M)$ of a K-algebra A with coefficients in a two-sided A-module M coincide with those defined by Hochschild [4] in the case when A is K-projective. Recently Azumaya showed in [1] the validity of the same fact under the weaker condition of the A^*-projectivity of A^e. We shall show in §2 that the Azumaya theorem can also be proved in the similar way as in Cartan and Eilenberg [2, IX, §6]. In §3 and §4, we shall give some results concerning projective dimensions of algebras and concerning supplemented algebras respectively, also under the condition of A^*-projectivity of A^e. Finally, we shall obtain in §5 a characterization of the Dedekind ring.

Throughout in this paper, we assume that a ring A considered has an identity element and all A-modules are unital, and we use always the notation $\otimes$ instead of $\otimes_K$.

§ 2. Let A be an associative algebra over a commutative ring K, and A^e the enveloping algebra of A: $A^e = A \otimes A^*$, where A* is the opposite algebra of A.

For each integer $n \geqq -1$, let $S_n(A)$ denote the $(n+2)$-fold tensor product over K of A with itself. Thus $S_{-1}(A) = A$, $S_{n+1}(A) = A \otimes S_n(A)$. We convert $S_n(A)$ into a left A^e-module by setting $(b \otimes c^*)(a_0 \otimes a_1 \otimes \cdots \otimes a_n \otimes a_{n+1}) = (ba_0) \otimes a_1 \otimes \cdots \otimes a_n \otimes (a_{n+1}c)$.

Lemma 1. *If the enveloping algebra A^e of A is projective as a left A^*-module, then, for $n \geqq 0$, $S_n(A)$ is projective as a left A^e-module.*

Proof. We shall prove this by induction on n. For $n=0$, this is evident since $S_0(A) = A \otimes A$ is isomorphic with $A^e = A \otimes A^*$ as a left A^e-module. Suppose now that we already know that $S_{n-1}(A)$ is A^e-projective. The left A^e-module $S_{n-1}(A)$ may be considered as a left A^*-module by setting