A CHARACTERIZATION OF UNIFORMLY DISTRIBUTED SEQUENCES OF INTEGERS

By

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1. Introduction. Let m be an integer not less than two. An infinite sequence $A=(a_n)$ of integers is said to be *uniformly distributed modulo* m, if the limit

$$\lim_{N\to\infty}\frac{1}{N}A(N, j, m)=\frac{1}{m}$$

exists for all $j=0, 1,\cdots, m-1$, where $A(N, j, m)$ is the number of terms a_n $(1\leqq n\leqq N)$ which are $\equiv j \pmod{m}$. If the sequence A is uniformly distributed modulo m for every integer $m\geqq 2$, then we say that A is *uniformly distributed.*

The notion of uniform distribution of sequences of integers, which is in a sense dual to that of uniform distribution (mod 1) of sequences of real numbers, is due to I. Niven [4], who obtained a number of interesting results on uniformly distributed sequences of integers. And a criterion for a sequence $A=(a_n)$ of integers should be uniformly distributed has been given by one of the present authors (see [5]): thus, a necessary and sufficient condition that the sequence A be uniformly distributed modulo m, where $m\geqq 2$, is that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\exp\left(2\pi i a_n\frac{h}{m}\right)=0$$

for all $h=1,\cdots, m-1$. Hence, the sequence A is uniformly distributed if and only if

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\exp(2\pi i a_n t)=0$$

for all rational numbers t with $t\not\equiv 0 \pmod{1}$.

The main purpose of this note is to present another characterization of uniformly distributed sequences of integers, making use of a kind of integrals defined over the space of integers. This, as well as the criterion quoted above, will have some analogy with the well-known characterization of uniform distribution (mod 1) of sequences of real numbers (cf. [3; Chap. IV]).