ON THE CONTINUITY AND THE MONOTONOUSNESS OF NORMS

By

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§ 1. Let R be a *universally continuous semi-ordered linear space*[1] (i.e. *conditionally complete vector lattice* in Birkhoff's sense) and $\|\cdot\|$ be a norm on R satisfying the following conditions throughout this paper:

(N. 1) $|x| \leqq |y|$ $(x, y \in R)$ *implies* $\|x\| \leqq \|y\|$;

(N. 2) $0 \leqq x_\lambda \uparrow_{\lambda \in \Lambda} x$ *implies* $\|x\| = \sup_{\lambda \in \Lambda} \|x_\lambda\|$[2].

A norm $\|\cdot\|$ on R is called *continuous*, if

$$\inf_{\nu=1,2,\cdots} \|x_\nu\| = 0 \qquad \text{for any } x_\nu \downarrow_{\nu=1}^{\infty} 0\text{[3]}. \tag{1.1}$$

The continuity of norms on R plays an important rôle in the theory of semi-ordered linear spaces. In fact, it is well known [8, 9; § 31] that *every norm-continuous linear functional f on R is (order-) universally continuous*, i.e.

$$\inf_{\lambda \in \Lambda} |f(x_\lambda)| = 0 \qquad \text{for any } x_\lambda \downarrow_{\lambda \in \Lambda} 0, \tag{1.2}$$

and R becomes *superuniversally continuous*[4] as a space in this case.

It is clear that if a norm $\|\cdot\|$ on R is continuous, the another norm $\|\cdot\|_1$ which is equivalent to $\|\cdot\|$ is also continuous. As for the conditions under which norms $\|\cdot\|$ on R are continuous, there are the detailed investigations by T. Andô [3, 4].

A norm $\|\cdot\|$ on R is called *monotone* [8], if

$$|x| \lneqq |y| \ (x, y \in R) \qquad \textit{implies } \|x\| \lneqq \|y\|, \tag{1.3}$$

and is called *uniformly monotone* [8, 9; § 30], if

1) This terminology is due to H. Nakano [9]. We use mainly notation and terminology of [9] here.

2) A norm satisfying (N. 1) and (N. 2) is called *semi-continuous* in [10]. A norm on $\|\cdot\|$ satisfying (N. 1) is called *monotone* in [7]. On the other hand, (N. 1) is assumed for any norm of normed lattices in [6].

3) This means $x_1 \geqq x_2 \geqq \cdots \geqq 0$ and $\bigcap_{\nu=1}^{\infty} x_\nu = 0$.

4) R is called *superuniversally continuous*, if for any $0 \leqq x_{\lambda, \lambda \in \Lambda} \leqq a$ there exists $\{x_{\lambda_\nu}\}_{\nu=1}^{\infty} \leqq \{x_\lambda\}_{\lambda \in \Lambda}$ such that $\bigcup_{\nu=1}^{\infty} x_{\lambda_\nu} = \bigcup_{\lambda \in \Lambda} x_\lambda$.