ON A DIFFERENTIAL-DIFFERENCE EQUATION

By

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In connexion with the study of certain incomplete sums of multiplicative functions, N. G. de Bruijn and J. H. van Lint [3] have introduced the function $f_s(x)$ $(s \geqq 0)$ satisfying the set of conditions:

(i) $f_s(x) = 0$ for $x<0$,

(ii) $f_s(x)$ is continuous for $x>0$,

(iii) $f_s(x) = x^{s-1}$ for $0<x\leqq 1$,

(iv) $xf_s'(x) = (s-1)f_s(x) - sf_s(x-1)$ for $x>1$.

(The function $f_s(x)$ is originally defined in [3; II, §2] only for $x>0$; it will be convenient, however, to define $f_s(x)=0$ for $x<0$ for our purpose.)

On the other hand, N. G. de Bruijn [1 and 2] has investigated in detail the property and behaviour of $f_s(x)$ for $s=1$. In particular, there he obtained an explicit formula for $f_1(x)$:

$$f_1(x) = \frac{e^C}{2\pi i}\int_{-i\infty}^{i\infty} \exp\left(-xt + \int_0^t \frac{e^z-1}{z}dz\right)dt \qquad (x>0),$$

where C is Euler's constant,

$$C = \lim_{n\to\infty}\left(\sum_{m=1}^{n}\frac{1}{m} - \log n\right).$$

In the present note we shall prove an analogous formula for $f_s(x)$ with general $s>0$.

Remark. For $s=0$ it is easy to see that $f_s(x)=f_0(x)=x^{-1}$ $(x>0)$. We may suppose, therefore, that $s>0$ throughout in the following.

1. Lemmata. We require two lemmas independent of one another.

Lemma 1. *If $\phi(s)$ is a (complex valued) continuous function defined for $s>0$ and satisfying the functional equation*

$$\phi(s+r) = \phi(s)\phi(r) \qquad (s>0,\ r>0),$$

then there is an integer A independent of s such that

$$\phi(s) = e^{2\pi i A s}\left(\phi(1)\right)^s \qquad (s>0).$$