On the geodesic projective transformation in Riemannian spaces

Dedicated to Professor Yoshie Katsurada on her sixtieth birthday

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Introduction. In a recent paper [2], the author introduced the notion of the geodesic conformal transformation and discussed Riemannian spaces which admit such transformations at each point. The geodesic conformal transformation at a point O is, roughly speaking, a local conformal transformation which leaves invariant any geodesic through O. The purpose of this paper is to give the projective analogy. Though the results are not satisfactory comparing to the conformal case, the geodesic projective transfomation seems to be interesting. Because, when we seek after the projective analogy of locally symmetric spaces, it must play a basic role.

1. Normal coordinates. Let M^n be an n dimensional analytic Riemannian space with positive definite metric g_{ij}[1]. Consider a normal coordinate $\{x^i\}$ of origin O in a normal neighbourhood U, then

$$\left\{ {h \atop ij} \right\} x^i x^j = 0 \tag{1.1}$$

hold good in U. Any geodesic γ through O is given by $x^i = \xi^i s$, where ξ^i is the unit tangent vector of γ at O and s means the arc length along γ. Throughout the paper we shall only use such a coodinate, and consider s to be positive, unless otherwise stated. f_i, f_{ij}, $\cdots$ mean the successive derivatives of f with respect to x^i, $x^j, \cdots$, and f', f'', $\cdots$ the ones with respect to s.

The following identities in U are well known.

$$(g_{ij})_0 x^i x^j = g_{ij} x^i x^j = s^2, \tag{1.2}$$

$$s_i x^i = s, \tag{1.3}$$

$$s_{ij} x^j = 0. \tag{1.4}$$

If we put $s^i = g^{ij} s_j$ and $x_i = (g_{ij})_0 x^j$, then

$$g_{ij} x^j = x_i, \tag{1.5}$$

$$g_{ij}(\rho x) x^i x^j = x_i x^i = s^2 \qquad \text{for any small } \rho, \tag{1.6}$$

1) We follow Yano-Bochner's notations [1].