On the commutator of differential operators[1)]

Dedicated to Prof. Yoshie Katsurada celebrating her sixtieth birthday

By Takeshi SUMITOMO

§ 0. **Introduction.** In [3][2)] A. Lichnorowicz has studied the basic properties of differential operators in several kinds of spaces such as the differentiable manifold with a torsionless connection, the reductive homogeneous space with an invariant volume element, and the symmetric space. Basing upon these fundamental researches he has reproved the known theorem due to I. M. Gelfand that the algebra of invariant differential operators on a globally symmetric Riemannian manifold be a commutative one. Such an algebra in fact has a structure of a polynomial ring and there are many applications of this theorem due to A. Selberg, H. Chandra and others in several branches of mathematics such as the theory of numbers, theory of spherical functions and modern physics.

In the present paper we try to study a rather converse problem of Gelfand's theorem basing upon the same foundations and formulas in [3]. To explain our situation more explicitely we propose the following problem: Have a Riemannian homogeneous space with the commutative algebra of invariant differential operators, a parallel Ricci tensor? As an incomplete answer to this problem one of the identities obtained in the present paper contains as a special case the following two identities which are valid under a suitable commutative condition,

$$\nabla_n(R_{ijkl}R_m{}^{jkl})+\nabla_i(R_{mjkl}R_n{}^{jkl})+\nabla_m(R_{njkl}R_i{}^{jkl})=0 \tag{0.1}$$

$$\nabla_k R_{ij}+\nabla_j R_{ki}+\nabla_i R_{jk}=0 \tag{0.2}$$

It is notable that the above identities consist, as a special case, in a weakly symmetric space introduced by A. Selberg [12]. Another remarkable result is that any harmonic vector field be a parallel one in a compact weakly symmetric space.

In § 1 terminologies, fundamental concepts and basic theorems about differential operators are given. In § 2 the commutators of differential operators on Riemannian manifolds are calculated explicitely. In § 3 we obtain

1) A resume of a part of this work is contained in [8].

2) Numbers in brackets refer to the references at the end of the paper.