Sufficient conditions for an almost-Hermitian manifold to be Kählerian

Dedicated to Professor Y. Katsurada on her 60th birthday

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§ 0. Introduction

If an almost-Hermitian manifold M is a Kählerian manifold, then its curvature tensor R satisfies

$$(*) \qquad R(X, Y)\cdot F=0 \qquad \text{for all tangent vectors } X \text{ and } Y,$$

where the endomorphism $R(X, Y)$ operates on the almost-complex structure tensor F as a derivation at each point on M.

Conversely, does this algebraic condition (*) on the almost-complex structure tensor field F imply that M is a Kählerian manifold? For an almost-Kählerian manifold or a K-space, Kotō and the present author (Sawaki and Kotō [3]) already showed that the answer is affirmative, that is,

THEOREM A. *If an almost-Kählerian manifold or a K-space M satisfies* $S=S^*$, *then M is Kählerian, where S is the scalar curvature and* $S^*=\frac{1}{2} F^{ab}R_{abt}{}^{c}F_{c}^{t}$.

In this theorem, the condition $S=S^*$ is weaker than $R(X, Y)\cdot F=0$, in fact, $R(X, Y)\cdot F=0$ implies $S=S^*$.

This problem for an almost-Kählerian manifold has also been studied recently by Goldberg [1] and under some additional conditions the present author [4] has proved the following

THEOREM B. *If an almost-Hermitian manifold M satisfies*

(i) $R(X, Y)\cdot F=0$, $R(X, Y)\cdot \nabla F=0$ *for all tangent vectors X and Y,*

(ii) $\nabla_{[j}S_{i]h}=0$ (*or equivalently* $\nabla_t R^t{}_{jih}=0$),

(iii) *the Ricci form is definite,*

then M is Kählerian.

THEOREM C. *If a compact Hermitian manifold M satisfies*

(i) $R(X, Y)\cdot F=0$, $R(X, Y)\cdot \Omega=0$ *for all tangent vectors X and Y,*[1)]

1) In the sequel, we omit "for all tangent vectors X and Y".