Riemannian manifolds admitting more than $n-1$ linearly independent solutions of $\nabla^2\rho+c^2\rho g=0$

Dedicated to Prof. Yoshie Katsurada on the occassion of her sixtieth birthday

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Let M be a connected C^∞-Riemannian manifold of $n\,(\geqq 2)$ dimensions with Riemannian metric g. Let us consider the system of partial differential equations

$$(1)\qquad \nabla_j\nabla_i\rho+c^2\rho g_{ji}=0 \qquad (c>0)$$

on M, where ∇_i denote the local components of the covariant derivative with respect to the Riemannian connection associated to g $(i, j, k, \cdots=1, 2, \cdots, n)$.

In a complete Riemannian manifold the existence of a non-trivial solution of (1) uniquely determines the Riemannian manifold structure up to an isometry. In fact, the following theorem is well known.

THEOREM A (Obata [1], [2], [3]). *Let M be complete. In order for M to admit a non-trivial solution of* (1), *it is necessary and sufficient that M be isometric to a sphere $S^n\left(\frac{1}{c}\right)$ of radius $\frac{1}{c}$ in the* $(n+1)$*-dimensional Euclidean space E^{n+1}.*

In the present paper we shall deal with Riemannian manifolds, admitting more than $n-1$ linearly independent solutions of (1), instead of the assumption of completeness, and prove the following three theorems.

THEOREM B. *In order for M to admit $n+1$ solutions of* (1), *linearly independent over the real number field R, it is necessary and sufficient that M be isometrically immersed in $S^n\left(\frac{1}{c}\right)$ in E^{n+1}.*

THEOREM C. *Let M be simply connected. In order for M to admit n solutions of* (1), *linearly independent over R, it is necessary and sufficient that M be isometrically immersed in $S^n\left(\frac{1}{c}\right)$* in E^{n+1}.

THEOREM D. *If M admit $n-1$ solutions of* (1), *linearly independent over R, M is of constant curvature c^2.*

The rest of the present paper is devoted itself to the proofs of these three theorems.