Normal subgroups of quadruply transitive permutation groups*

To Yoshie Katsurada on her Sixtieth Birthday

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Introduction. Let Ω be the set of symbols $1, \cdots, n$. Let G be a permutation group on Ω. Wagner [6] proved the following theorem:

If G is triply transitive and if n is odd and greater tnan 3, then every normal subgroup $(\neq 1)$ of G is also triply transitive.

In this note we prove the following theorem:

THEOREM. *If G is quadruply transitive and if n is prime to 3 and greater than 5, then every normal subgroup $(\neq 1)$ of G is also quadruply transitive.*

The outline of the proof is as follows. First of all, by the above theorem of Wagner we may assume that n is odd. Let H $(\neq 1)$ be a normal subgroup of G which is not quadruply transitive. Then without the restriction that n is prime to 3 we obtain some permutation-character theoretical results on H which are slightly more than needed in the proof. At the final point, with the restriction that n is prime to 3 we utilize results obtained above to get a contradiction.

Definitions and Notation. Let $x \in G$. Then $\alpha(x)$, $\beta(x)$, $\gamma(x)$ and $\delta(x)$ denote the numbers of 1-, 2-, 3- and 4-cycles in the permutation structure of x respectively. Let $X \subseteq G$. Then $\alpha(X)$ denotes the set of symbols of Ω each of which is fixed by X. Let X be a subgroup of G. Let φ and ψ be class functions on X. Then $(\varphi, \psi)_X = \frac{1}{|X|} \sum_{x \in X} \varphi(x)\overline{\psi(x)}$ and $N_X(\varphi) = (\varphi, \varphi)_X$. $X_{(\Delta)}$ and X_Δ denote the global and pointwise stabilizers of Δ in X respectively. $X_{(\Delta)}^{\Delta}$ denotes the restriction of $X_{(\Delta)}$ to Δ. If $\Delta = \{1\}$, $\{1, 2\}$ or $\{1, 2, 3\}$, we also write X_1, $X_{1,2}$ or $X_{1,2,3}$ instead of X_Δ. Let Y be a subgroup of X. Then $Ns_X Y$ denotes the normalizer of Y in X. $LF(2, q)$ denotes the linear fractional group over the field of q elements.

PROOF. (a) The following permutation-character theoretical formulae for quadruply (and triply) transitive permutation groups are well-known ([4], p. 597; [7], (9.9)).

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