3-dimensional Riemannian manifolds satisfying $R(X, Y)\cdot R=0$

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§ 1. Introduction

Let (M, g) be a Riemannian manifold with a positive definite metric tensor g. By R we denote the Riemannian curvature tensor. By M_p we denote the tangent space to M at p. Let $X, Y\in M_p$. Then $R(X, Y)$ operates on the tensor algebra as a derivation at each point p. In a locally symmetric space (i.e., $\nabla R=0$), we have $R(X, Y)\cdot R=0$. We consider the converse under some additional conditions.

THEOREM. *Let (M, g) be a complete and irreducible 3-dimensional Riemannian manifold. Assume that the scalar curvature S is positive and bounded away from zero (i.e., $S\geq\varepsilon>0$ for some constant ε). If (M, g) satisfies*

(*) $R(X, Y)\cdot R=0$ *for any $p\in M$ and $X, Y\in M_p$,*

then (M, g) is of positive constant curvature.

This theorem follows from the following

PROPOSITION. *Let (M, g) be a complete 3-dimensional Riemannian manifold satisfying (*). Assume that S is positive and bounded away from zero. Then (M, g) is either*

(1) *a space of positive constant curvature, or*

(2) *locally a product Riemannian manifold of a 2-dimensional space of positive curvature and a real line.*

A consequence of Theorem is as follows:

COROLLARY. *Let (M, g) be a compact and irreducible 3-dimensional Riemannian manifold. If (M, g) satisfies (*) and S is positive, then (M, g) is of positive constant curvature.*

In Theorem the condition on the scalar curvature or something like this is necessary, because of Takagi's example [6].

It may be noticed that (*) is equivalent to $R(X, Y)\cdot R_1=0$, where R_1 denotes the Ricci curvature tensor. In this paper (M, g) is assumed to be connected and of class C^∞.

§ 2. Preliminaries

Let (M, g) be a 3-dimensional Riemannian manifold and assume (*) on