On a problem of D. G. Higman

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Dedicated to Professor Kiiti Morita on his 60th birthday

In his paper [3], D. G. Higman gave a characterization of (projective) symplectic groups $PS_p(4, q)$ of dimension 4 over the field F_q ([3], Theorem 2) and proposed the similar characterization for higher dimensional case. In this note, we will give a characterization of higher dimensional symplectic groups by adopting Kantor's idea in [5].

For notation we follow that of Higman [3] mostly. Given a group G of permutations of a finite set Ω we denote by a^g the image of $a\in\Omega$ under $g\in G$, and by G_a the stabilizer of a, $G_a=\{g\in G|a^g=a\}$. For a subgroup H of G and a subset X of Ω we let $a^H=\{a^g|g\in H\}$, $X^g=\{a^g|a\in X\}$ and $G_X=\bigcap_{a\in X} G_a$. We call the number of orbits of G_a, $a\in\Omega$, the rank of G and we call the lengths of these orbits the subdegrees of G. Our theorem is the following.

THEOREM. *Let G be a transitive rank 3 permutation group on a finite set Ω whose subdegrees are 1, $(q^{n-1}-q)/(q-1)$, q^{n-1} where q is a power of a prime number p and $n\geqq 4$. Assume that there are at least q elements of G_a, $a\in\Omega$, fixing a G_a-orbit of length $(q^{n-1}-q)/(q-1)$ pointwise. Then n is even and G contains a normal subgroup isomorphic to the projective symplectic group $PS_p(n, q)$ which is generated by all the symplectic elations.*

Proof. For $a\in\Omega$, we denote G_a-orbits by $\{a\}, \Delta(a), \Gamma(a)$ with $\Delta(a)^g=\Delta(a^g)$, $\Gamma(a)^g=\Gamma(a^g)\,(g\in G)$ and $|\Delta(a)|=(q^{n-1}-q)/(q-1)$, $|\Gamma(a)|=q^{n-1}$. The intersection numbers λ, μ of G are defined by

$$|\Delta(a)\cap\Delta(b)|=\begin{cases}\lambda & \text{if } b\in\Delta(a)\\ \mu & \text{if } b\in\Gamma(a)\,.\end{cases}$$

Aecording to Lemma 5 in [3], we have

$$\mu q^{n-1}=\frac{q^{n-1}-q}{q-1}\left(\frac{q^{n-1}-q}{q-1}-\lambda-1\right).$$

Hence $\mu=1+q+\cdots+q^{n-3}$ and $\lambda=-1+q+\cdots+q^{n-3}$. Thus, by Lemma 8 in [3], a block design $\mathscr{D}$ whose points are the elements of Ω and whose blocks are the symbols $b^{\perp}$, one for each $b\in\Omega$, and whose incidence $a\in b^{\perp}$