Note on H-separable extensions

Dedicated to Professor Kiiti Morita on his 60th birthday

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It is the purpose of this note to give a (self-contained) computational proof to the principal theorem (1. 3) of [7] and a theorem concerning ring-endomorphisms of an H-separable extension. Our tool employed in this note is an H-system of an H-separable extension, which was introduced in [4].

Recently, we found that the proof of [6, Proposition 1] contained an error and the same was repeated in the proof of main part of [7, (1. 3)]. So, the present note comprehends the correction to the previous papers [6] and [7].

Throughout, A/B will represent a ring extension with common identity 1, V the centralizer $V_A(B)$ of B in A, and C the center of A.

The next will be useful occasionally in the subsequent study.

(1) *Let $B'\subset B''$ be intermediate rings of A/B. Let $V'=V_A(B')$, and $V''=V_A(B'')$. If ${}_{B'}B'\otimes_B B''_{B''}\to {}_{B'}B''_{B''}(b'\otimes b''\mapsto b'b'')$ splits then ${}_{V'}V'_{V''}<\oplus_{V'}V_{V''}$.*

PROOF. There exists an element $\sum_k b'_k\otimes b''_k\in(B'\otimes_B B'')^{B'}$ such that $\sum_k b'_k b''_k=1$. Then, the map $q: V\to V'$ defined by $v\mapsto\sum_k b'_k v b''_k$ is a V'-V''-homomorphism and induces the identity map on V', which means ${}_{V'}V'_{V''}<\oplus_{V'}V_{V''}$.

A/B is called an *H-separable extension* if $A\otimes_B A$ is A-A-isomorphic to an A-A-direct summand of a finite direct sum of copies of A. To be easily seen, A/B is H-separable if and only if there exist some $v_i\in V$ $(i=1,\cdots,m)$ and $\sum_j x_{ij}\otimes y_{ij}\in(A\otimes_B A)^A$ such that $\sum_{i,j}x_{ij}\otimes y_{ij}v_i=1\otimes 1$. Following [4], such a system $\{v_i\,;\,\sum_j x_{ij}\otimes y_{ij}\}_i$ will be called an *H-system* for A/B.

In what follows, we assume always A/B is an H-separable extension with an H-system $\{v_i\,;\,\sum_j x_{ij}\otimes y_{ij}\}_i$. Then the map $\eta: A\otimes_B A\to \mathrm{Hom}_C(V,A)$ $(a_1\otimes a_2\mapsto(v\mapsto a_1va_2))$ is an A-A-isomorphism, whose inverse is given by $h\mapsto\sum_{i,j}x_{ij}\otimes y_{ij}h(v_i)$ (cf. also (2. 1')).

(2) *If σ is an arbitrary ring-endomorphism of A which leaves every element of B invariant, $g\in\mathrm{Hom}(A_B, A_B)$ and $h\in\mathrm{Hom}({}_BA, {}_BA)$, then*