The endomorphism ring of an indecomposable module with an Artinian projective cover

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As is well known, each endomorphism of an Artinian, Noetherian and indecomposable module is either nilpotent or an automorphism. C. I. Vinsonhaler has proved that each endomorphism of an indecomposable module with a Noetherian injective hull is either nilpotent or an automorphism (see [6], [7] and [5 ; p. 75]). In this note we shall verify that each endomorphism of an indecomposable module with an Artinian projective cover is either nilpotent or an automorphism.

We will assume throughout that M denotes a nonzero unital left R-module, where R is a nonzero ring with identity.

THEOREM. *Let M be an indecomposable left R-module and let P be an Artinian left R-module which is a projective cover of M with a minimal epimorphism $\varphi: P \rightarrow M$. Then each endomorphism α of M is either nilpotent or an automorphism.*

REMARK 1. M is called a semiperfect module iff every factor module of M has a projective cover (cf. Mares [4]).

Let N be a submodule of M. A cocomplement N^c of N in M is a minimal submodule of M such that $N + N^c = M$. M is called a cocomplemented module iff every submodule of M has a cocomplement in M.

As is easily proven, semiperfect modules are cocomplemented. Conversely, projective cocomplemented modules are semiperfect (see Kasch-Mares [3]).

REMARK 2. In the following commutative diagram with left R-modules M_i and homomorphisms ψ_i $(i=1, 2, 3)$:

$$\begin{tikzcd} & M_1 \arrow[dl, "\psi_3"'] \arrow[d, "\psi_1"] \\ M_3 \arrow[r, "\psi_2"] & M_2, \end{tikzcd}$$

if ψ_1 is an epimorphism and if ψ_2 is a minimal epimorphism, then ψ_3 is necessarily an epimorphism.