On the index theorem of Ambrose

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1. Introduction

The index theorem for geodesics under the general boundary condition (two variable end points) has been given by W. Ambrose ([1], see also T. Takahashi [5]). But his proof is very complicated. M. Klingmann ([4]) proved the somewhat more general index theorem using the theory of quadratic forms on Hilbert space. Recently W. Klingenberg ([2], [3]) has obtained the index theorem for closed geodesics from the geodesic flow view point. The purpose of the present note is to give another simple proof of the Ambrose index theorem via Klingenberg's view point. In fact, we need only the fundamental properties of Jacobi fields. Since the concept of conjugate point defined in [1] is not so familiar, we shall give the explicit statement of the Ambrose index theorem for completeness.

Let $(M, \langle\, ,\rangle)$ be a riemannian manifold and K, L be submanifolds of M. Let $c: [a, b]\to M$ be a normal geodesic such that $c(a)\in K$, $c(b)\in L$, $\dot{c}(a)\perp T_{c(a)}K$, $\dot{c}(b)\perp T_{c(b)}L$, where $T_{c(a)}K$ etc. denotes the tangent space to K at $c(a)$. We will be concerned with the "number of essentially different curves connecting K and L which are shorter than c". First we shall give some preliminaries.

1.1. Boundary conditions. A boundary condition at $t(a\leqq t\leqq b)$ is, by definition, a pair $\mathscr{S}=(S, A_S)$ where S is a subspace of $\perp\dot{c}(t)$ (the orthogonal complement of $\dot{c}(t)$ in $T_{c(t)}M$) and $A_S: S\to S$ is a self-adjoint linear mapping of S.

EXAMPLE 1. Let P be a submanifold of M which is perpendicular to c at $c(t)$. Then we have the boundary condition (S, A_S) at t by $S:=T_{c(t)}P$, $\langle A_S X, Y\rangle := H_{\dot{c}(t)}(X, Y)$, where $H_{\dot{c}(t)}$ denotes the second fundamental form of P relative to the normal $\dot{c}(t)$.

Let $\mathscr{J}$ be a vector space of Jacobi fields along c which is perpendicular to c. We shall denote the covariant differentiation with respect to $\dot{c}(t)$ by ∇. If the boundary condition $\mathscr{S}$ at t is given, we define

$$\mathscr{J}_S^* := \left\{Y\in\mathscr{J} \mid Y(t)\in S,\quad \nabla Y(t)-A_S Y(t)\perp S\right\}.\ \dim \mathscr{J}_S^*=\dim M-1.$$

$$\mathscr{J}_S := \left\{Y\in\mathscr{J} \mid Y(t)\in S,\quad \nabla Y(t)= A_S Y(t)\right\}.\ \dim \mathscr{J}_S=\dim S.$$