Some remarks on nonlinear differential equations in Banach spaces

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§ 1. Introduction and results.

Let E be a Banach space with the dual space E^*. The norms in E and E^* are denoted by $\|\|$. We denote by $S(u, r)$ the closed sphere of center u with radius r.

In this paper we are concerned with nonlinear abstract Cauchy problems of the forms

$$(D_1) \qquad \frac{d}{dt} u(t) = f(t, u(t)), \qquad u(0) = u_0 \in E,$$

and

$$(D_2) \qquad \frac{d}{dt} u(t) = Au(t) + f(t, u(t)), \qquad u(0) = u_0 \in D(A).$$

Here A is a nonlinear operator with domain $D(A)$ and range $R(A)$ in E, and f is a E-valued mapping defined on $[0, T] \times S(u_0, r)$ or on $[0, \infty) \times E$.

It is well known that in the case of $E = R^n$, the n-dimensional Euclidean space, the continuity of f in a neighbourhood of $(0, u_0)$ alone implies the existence of a local solution of (D_1). This is the classical Peano's theorem. However, this theorem cannot be generalized to the infinite-dimensional case (see [3], [16]).

It is our object in this paper to give sufficient conditions for the existence of the unique solutions to the Cauchy problems of the forms (D_1) and (D_2).

Let the functionals $\langle , \rangle_1$ and $\langle , \rangle_2$ be defined as follows (cf. M. Hasegawa [6]):

$$\langle u, v \rangle_1 = \lim_{h \downarrow 0} \frac{1}{h} (\|u + hv\| - \|u\|),$$

and

$$\langle u, v \rangle_2 = \frac{1}{2} (\langle u, v \rangle_1 - \langle u, -v \rangle_1)$$

for u, v in E.

In order to prove the existence of the unique solution of the equation