Almost immersions

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1. We work in the category of compact PL spaces and PL maps [Z]. Given spaces X and Y and a map $f: X \to Y$, we define $U_r(f)$ to be the set of points $x \in X$ such that $f^{-1}f(x)$ contains at least r points. $S_r(f)$ is the closure of $U_r(f)$ in X. We call $B(f)=S_2(f)-U_2(f)$ branch locus of f (i.e. for $x \in B(f)$ $f^{-1}f(x)=x$ but $f|U(x)$ is not embedding for any neighborhood $U(x)$ of x.) Conversely we consider whether $f|U_2(f)$ is a local embedding (=immersion) or not (i.e. for $x \in U_2(f)$ there is a neighborhood $U(x)$ of x such that $f|U(x)$ is an embedding). For this problem we obtain a following.

PROPOSITION. 1. *If M^m, Q^q are closed m- and q-*dim. *manifolds and if $f: M^m \to Q^q$ $(m<q)$ is a map, $f|U_2(f)$ is an immersion.*

Next we consider about almost immersions. Given a map $f: M \to Q$, we call f almost immersion if $B(f)=S_2(f)-U_2(f)=$one point and we call f special almost immersion if f is an almost immersion and if $S_2(f|lk(a, M))=\{a_1, \cdots, a_{2n}\}$ (finite set of even points) for a unique point $\{a\}=B(f)$ and $S_3(f|lk(a, M))=\phi$. According to Irwin [1] f is a simple immersion if $S_3(f)=\phi=B(f)$. Then we prove

THEOREM 1. *Let M^m, Q^q be $(2m-q-1)$-connected closed manifolds, $3m \leqq 2q-1$ and $q \geqq 6$. Then any PL map $f: M \to Q$ is homotopic to a special almost immersion g; $M \to Q$ with $S_3(g)=\phi$.*

THEOREM 2. *Let M^m be $(2m-q-1)$-connected closed manifold and Q^q be $(2\mathrm{m}-q)$-connected, $q \leqq m-3$. Then any PL map f; $M \to Q$ is homotopic to an almost immersion $g: M \to Q$ with $S_3(g)=\phi$.*

In this paper M, Q always mean m- and q-dim. closed manifold if otherwise is not stated. D^n, S^n are always n-dim. ball and sphere respectively. For a simplial complex K and a simplex Δ^t in K, let

$$St(\Delta^t, K)=\left\{\Delta^p \in K \,\middle|\, \Delta^p<\Delta^q, \Delta^q>\Delta^t\right\},$$
$$Lk(\Delta^t, K)=\left\{\Delta^p \in St(\Delta^t, K) \,\middle|\, \Delta^p \cap \Delta^t=\phi\right\} \text{ be}$$

the star and the link of Δ^t in K. For spaces or complexes X and Y, $X*Y$ denote a non-singular join of X with Y. We call $X*Y$ a linear cone on Y if X is one point. ∂M, $\operatorname{Int} M$ mean the boundary, the interior of the