On cofinite-dimensional modules

Dedicated to Professor Kiiti Morita on his sixtieth birthday

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Introduction

Goldie introduced finite-dimensional modules in [4]. By dualizing the notion of finite-dimensionality, "cofinite-dimensional modules" may be defind. The object of this article is to study the properties of cofinite-dimensional modules under certain conditions. Our basic tools are coessential extensions and cocomplements in a module, and our main guides are Miyashita [9], [10] and Utumi [14].

It will be assumed throughout that R is a nonzero ring with identity and that all modules over R are unital left R-modules. Let M be a nonzero R-module and let $A\subset B$ be submodules[1] of M. Then B is called a coessential extension of A in M iff B/A is a small submodule of M/A. This definition originates in the necessity of treating not merely small submodules of M but small submodules of factor modules of M. A set $\{A_\lambda|\lambda\in\Lambda\}$ of submodules of M is called coindependent iff $\bigcap_{i=1}^{n-1} A_{\lambda_i}+A_{\lambda_n}=M$ for any finite subset $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ of Λ $(n\geqq 2)$, and M is called cofinite-dimensional iff every coindependent set of submodules of M is finite. Zelinsky proves in [17] that every linearly compact module is cofinite-dimensional. As for the coindependency, Proposition 1.3 is fundamental and Proposition 1.6 shows the relationship between coessential extensions and coindependent sets of submodules.

For a submodule A of an R-module M, a complement A' of A in M is a maximal submodule of M with respect to the property $A\cap A'=0$; dually, a cocomplement A^c of A in M is a minimal submodule of M with respect to the property $A+A^c=M$. Clearly, each direct summand of M is a complement and also a cocomplement (of some submodule) in M. Section 2 is devoted to the propositions about cocomplements in a module.

It is proved by applying Zorn's Lemma that every submodule has a

1) Henceforward, submodules, factor modules, homomorphisms, epimorphisms, etc. of left R-modules will be understood to possess the sense of "R-".