Characterization of the p-conformally flat Riemannian manifold

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§1. Introduction.

Recently, Bang-yen Chen and Kentaro Yano [1] proved the following:

THEOREM 1. *In order that a Riemannian manifold M of dimension $n>3$ is conformally flat, it is necessary and sufficient that there exists a (unique) quadratic form Q on M such that the sectional curvature $K(\sigma)$ with respect to a plane σ is the trace of the restriction of Q to σ, i.e. $K(\sigma)=$ trace Q/σ, the metric being also restricted to σ.*

The object of this paper is to give the generalization of this theorem, *ipso facto*, the characterization of higher order conformally flatness.

We have the following:

THEOREM 2. *In order that a Riemannian manifold M of dimension $n\geqq 4p$ is p-conformally flat, it is necessary and sufficient that there exists a (unique) quadratic form Q, which satisfies the generalized first Bianchi identity as double form of type $(2p-1, 2p-1)$, on the bundle $\Lambda^{2p-1}(M)$ of $(2p-1)$-vectors of M such that the $2p$-th sectional curvature $K_{2p}(\sigma)$ with respect to an $2p$-plane σ is the trace of the restriction of Q to $\Lambda^{2p-1}(\sigma)$, i.e. $K_{2p}(\sigma)=$ trace $Q/\Lambda^{2p-1}(\sigma)$.*

§2. Preliminaries.

Let M be an n-dimensional Riemannian manifold with the Riemannian metric g, let $\mathfrak{F}(M)$ be the algebra of functions on M and let $\mathfrak{X}(M)$ be the Lie algebra of vector fields on M. In what follows we write $g=\langle\,,\rangle$, where it is convenient.

For p an integer between 1 and n, let $\Lambda^p(M)$ denote the bundle of p-vectors of M and let $\Lambda^p(m)$ be the fiber over $m\in M$. $\Lambda^p(M)$ is a Riemannian vector bundle, with the inner product on the fiber $\Lambda^p(m)$ over m related to the inner product on the tangent space M_m of M at m by

$$(2.1)\qquad \langle X_1\wedge X_2\wedge\cdots\wedge X_p,\ Y_1\wedge Y_2\wedge\cdots\wedge Y_p\rangle=\det\left[\langle X_i, Y_j\rangle\right],\ (X_i, Y_j\in M_m).$$

We define a double form of type (p, q) on M to be an $\mathfrak{F}(M)$-multilinear map