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A remark on the index of G-manifolds in the representation theory

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1. Introduction

Fix a finite group G. Let $\mathscr{F}$ be a family of subgroups of G which satisfies if $H\in\mathscr{F}$ and $H'\subset H$, then $H'\in\mathscr{F}$. Then G-bordism group of G-manifolds is denoted by $\Omega_{*}(G, \mathscr{F})$. And its elements are the bordism classes $[G, M]$ where M is a differentiable closed manifold and all isotropy groups G_x are in $\mathscr{F}$. Now we consider the index of G-manifolds. It is well known that the index I is a bordism invariant of Ω_{4k}. And it is extended naturally to the G-bordism invariant: $I: \Omega_{4k}(G, \mathscr{F})\longrightarrow RO(G)$, where $RO(G)$ is the Grothendieck group of G over R.

In this paper we compute the index of G-manifolds with $\mathscr{F}=\{1\}$ in $RO(G)$ in the sense of R. Lee [5].

2. The homomorphism I: $\Omega_{4k}(G, \mathscr{F})\longrightarrow RO(G)$

Let M be a compact oriented differentiable G-manifold without boundary and $\mathscr{F}$-free. The bilinear form $\Phi: H^{2k}(M; R)\times H^{2k}(M; R)\longrightarrow R$ is defined by $\Phi(x, y)=\langle x\cup y, [M]\rangle$, where $[M]$ is the orientation class of M. Then by the Poincaré duality, Φ is non-singular, symmetric and G-invariant. In $H^{2k}(M; R)$, we set G-invariant maximal subspaces

$$V_{+}=\left\{x\in H^{2k}(M; R)\,\middle|\,\Phi(x, x)>0 \text{ if } x\neq 0\right\}$$

$$V_{-}=\left\{x\in H^{2k}(M; R)\,\middle|\,\Phi(x, x)>0 \text{ if } x\neq 0\right\}, \quad \text{then}$$

$I: \Omega_{4k}(G, \mathscr{F})\longrightarrow RO(G)$ is defined by $I[G, M]=[V_{+}]-[V_{-}]$ (see [4] pp. 578), where $[V_{\pm}]$ is the equivalence class of $V_{\pm}$ in $RO(G)$. Now by the well known result (see [4] pp. 85–86), it is proved that

(2.1) The correspondence $I: \Omega_{4k}(G, \mathscr{F})\longrightarrow RO(G)$ is the well-defined homomorphism.

In particular, $G=\{1\}$, since $\Omega_{4k}(G, \mathscr{F})=\Omega_{4k}$ and $RO(G)=Z[K]$, where K is a trivial representation, $I: \Omega_{4k}\longrightarrow RO(G)$ is $I[M]=I(M)[K]$, where $I(M)$ is the index of M.