

On the torsion theoretic support of a module

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Goldman in [8] studied modules ${}_R X$ for which the localization X_σ has finite length in the quotient category determined by a torsion radical σ . He showed that for this to occur a necessary requirement is that the set of prime torsion radicals π such that $X_\pi \neq 0$ has only finitely many maximal elements; it is an open question whether this condition holds for a finitely generated module over any Noetherian ring. It was shown in [2] that the condition holds for the module ${}_R R$ if R is a ring with Krull dimension. This result will be used in section 1 below to show that any finitely generated module over a fully bounded, Noetherian ring has only finitely many maximal elements in its support. It will also be used to show that if R is a ring with Krull dimension, then R is Artinian if and only if every prime torsion radical is maximal, extending Theorem 5.10 of [8].

For a finitely generated module X over a commutative, Noetherian ring R , it is well-known that the following conditions hold: (i) a prime ideal belongs to the support of X if and only if it contains an associated prime ideal of X ; (ii) for any multiplicative set S of R , the associated prime ideals of the localization X_S correspond to the associated prime ideals of X which do not meet S ; (iii) for any multiplicative set S , if X is S -torsion, then so is the injective envelope $E(X)$ of X . Fully bounded modules are defined in section 2, and then in section 3 it is shown that any finitely generated, fully bounded, Artin-Rees module over a Noetherian ring satisfies the above conditions, with multiplicative sets replaced by torsion radicals and prime ideals replaced by prime torsion radicals. Cahen has shown in [4] that every module over a Noetherian ring satisfies the torsion theoretic form of condition (iii) if and only if every finitely generated module satisfies condition (i). This can be generalized to show that (with appropriate formulations) conditions (i), (ii) and (iii) are equivalent for any fixed module over a ring with Krull dimension. As Golan has observed in [7], any finite dimensional module which satisfies condition (i) must have only finitely many maximal elements in its support, since it has only finitely many associated primes.

Throughout the paper, R will denote an associative ring with identity,