Hokkaido Mathematical Journal Vol. 6 (1977) p. 302-305

On Harnack's pseudo-distance

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(Received December 17, 1976)

1. Introduction and terminology

In this paper we shall give a sufficient condition for which the Harnack's pseudo-distance h_R (denoted by d_R in [5]) on an arbitrary Riemann surface R is a real distance and we shall investigate a relation among the Harnack's pseudo-distance h_R, the Kobayashi's distance d_R ([4]) and the Carathéodory's distance c_R (cf. [4]).

For an arbitrary (open or closed) Riemann surface R, we denote by $HP=HP(R)$ the family of all positive harmonic functions on R. For any $a, b\in R$, we set

$$k_R(a, b) = \inf\left\{c\,;\ c^{-1}u(a)\leqq u(b)\leqq cu(a)\quad\text{for any}\quad u\in HP(R)\right\}$$

(Harnack's constant).

It is easy to see that $1\leqq k_R(a, b)<\infty$ and $(a, b)\rightarrow k_R(a, b)$ is continuous. Furthermore the following properties are easy to see:

$$k_R(a, b)=k_R(b, a)\quad\text{and}\quad k_R(a, b)\leqq k_R(a, c)\,k_R(c, b)\,.$$

The following definition is due to J. Köhn (cf. [2]).

DEFINITION 1. For any $a, b\in R$, we set $h_R(a, b)=\log\ k_R(a, b)$.

By definition, we see that $(a, b)\rightarrow h_R(a, b)$ is continuous and $R\in O_{HP}$ if and only if $h_R=0$. Furthermore h_R is a (real) distance if and only if $HP(R)$ separates points of R, i. e., for any $a, b\in R$ $(a\neq b)$, we can find $u\in HP(R)$ with $u(a)\neq u(b)$.

The following theorem is due to J. Köhn [5].

THEOREM 1 (An extension of Schwartz-Pick's theorem). *Let R, R' be two open Riemann surfaces. If f is an analytic mapping of R into R', then*

$$h_R(a, b)\geqq h_{R'}\left(f(a), f(b)\right)\quad \textit{for any } a, b\in R\,.$$

In particular, if $HP(R)$ separates points of R and R' is hyperbolic, then the equality holds if and only if f is an onto conformal mapping.

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