Hokkaido Mathematical Journal Vol. 6 (1977) p. 216-229

On some 4-dimensional Riemannian manifolds satisfying $R(X, Y)\cdot R=0$

By Kouei SEKIGAWA

(Received October 2, 1976)

1. Introduction.

Let (M, g) be a Riemannian manifold and ∇ be the Riemannian connection. By $T_x(M)$ and Exp_x we denote the tangent space to M at x and the exponential mapping of (M, g) at x. And let R, R_1 and S be the curvature tensor, the Ricci tensor and the scalar curvature of (M, g), respectively. For $X, Y\in T_x(M)$, $R(X, Y)$ operates on the tensor algebra as a derivation at each point $x\in M$. In a locally symmetric space ($\nabla R=0$), we have

(*) $\quad R(X, Y)\cdot R=0$ for any point $x\in M$ and $X, Y\in T_x(M)$.

We consider the converse under some additional conditions. If this paper, with respect to this problem, we shall prove

THEOREM A. *Let (M, g) be a complete, irreducible 4-dimensional Riemannian manifold of class C^∞ satisfying* (*). *If the volume of (M, g) is finite, then (M, g) is locally symmetric.*

COROLLARY A. *Let (M, g) be a compact, irreducible 4-dimensional Riemannian manifold of class C^∞ satisfying* (*). *Then (M, g) is locally symmetric.*

Let R^1 be the Ricci transformation defined by $g(R^1X, Y)=R_1(X, Y)$. In this paper, (M, g) is assumed to be connected, complete and of cleass C^∞ unless otherwise specified.

2. Lemmas.

Let (M, g) be a 4-dimensional complete Riemannian manifold. Assume (*). Let $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ be the eigenvalues of the Ricci transformation R^1 at a point $x\in M$. Then, we may have only the following five cases:

(I) $\lambda_1=\lambda_2=\lambda_3=\lambda_4=\lambda, \quad \lambda\neq 0,$

(II) $\lambda_1=\lambda_2=\lambda, \quad \lambda_3=\lambda_4=\mu, \quad \lambda\neq\mu, \quad \lambda, \mu\neq 0,$

(III) $\lambda_1=\lambda_2=\lambda_3=\lambda, \quad \lambda_4=0, \quad \lambda\neq 0,$

(IV) $\lambda_1=\lambda_2=\lambda, \quad \lambda_3=\lambda_4=0, \quad \lambda\neq 0,$