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Codominant dimensions and Morita equivalences

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Introduction

Let ${}_RP$ be a projective left R-module with endomorphism ring S. Let A be a left R-module. We say that P-codominant dimension of A is $\geqq n$, denoted by P-codom. dim. $A\geqq n$, if there exists an exact sequence:

$$X_n\longrightarrow X_{n-1}\longrightarrow\cdots\longrightarrow X_2\longrightarrow X_1\longrightarrow A\longrightarrow 0$$

where X_i's are isomorphic to direct sums of P's.

It is clear that P-codom. dim. $A\geqq 1$ iff P generates A. It is also equivalent with the condition $TA=A$, where T is the trace ideal of ${}_RP$. In this paper it is shown that P-codom. dim. $A\geqq 2$ iff $P\underset{S}{\otimes}\mathrm{Hom}_R(P, A)$ and A are canonically isomorphic. Another some equivalent conditions for this are also obtained in § 2. On the other hand, let ${}_SB$ be a left S-module. Then it is shown that B and $\mathrm{Hom}_R(P, P\underset{S}{\otimes}B)$ are canonically isomorphic iff $\mathrm{Hom}_R(P, Q)$-dom. dim. $B\geqq 2$, where ${}_RQ$ is an injective cogenerator in ${}_R\mathfrak{M}$. Thus we see that the categories $\mathscr{C}_1=\{X\in{}_R\mathfrak{M}|P\text{-codom. dim. }X\geqq 2\}$ and $\mathscr{C}_2=\{Y\in{}_S\mathfrak{M}|\mathrm{Hom}_R(P, Q)\text{-dom. dim. }Y\geqq 2\}$ are (canonically) equivalent. In case where ${}_RP$ is a progenerator in ${}_R\mathfrak{M}$, we have $\mathscr{C}_1={}_R\mathfrak{M}$ and, since ${}_S\mathrm{Hom}_R(P, Q)$ is an injective cogenerator in ${}_S\mathfrak{M}$, $\mathscr{C}_2={}_S\mathfrak{M}$. Thus our result affords a generalization of Morita equivalence. Another variations of an equivalence of this type are also discussed in § 1 and § 4.

Since the trace ideal T of a projective module ${}_RP$ is an idempotent two-sided ideal of R, T induce a torsion theory $(\mathscr{T}, \mathscr{F})$ in the category of left R-modules: $\mathscr{T}=\{X\in{}_R\mathfrak{M}|TX=X$, or equivalently, P-codom. dim. $X\geqq 1\}$, $\mathscr{F}=\{X'\in{}_R\mathfrak{M}|TX'=0\}$. The condition under which $(\mathscr{T}, \mathscr{F})$ is hereditary, that is, $\mathscr{T}$ is closed under submodules were studied recently by some authors ([1], [6]). Here we add some other conditions for this in § 3. Some of them are the followings:

(1) *The class* $\{X\in{}_R\mathfrak{M}|P\text{-codom. dim. }X\geqq 1\}$ *coincides with the class* $\{X'\in{}_R\mathfrak{M}|P\text{-codom. dim. }X'\geqq 2\}$.

(2) $P\underset{S}{\otimes}\mathrm{Hom}_R(P, X)$ *and* TX *are canonically isomorphic for every left*