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On modules which are flat over their endomorphism rings

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Let ${}_RM$ be a left R-module over a ring R[1], and S be the endomorphism ring of ${}_RM$. Let ${}_RA$ be a left R-module. We say that M-codominant dimension of A is $\geqq n$, if there is an exact sequence:

$$X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_2 \longrightarrow X_1 \longrightarrow A \longrightarrow 0\,,$$

where each X_i is isomorphic to a (finite or infinite) direct sum of copies of ${}_RM$. We denote by $\mathscr{C}_n$ the category of left R-modules whose M-codominant dimensions are $\geqq n$.

Recently T. Würfel has shown that, for a left R-module ${}_RM$, the following statements are equivalent:[2]

(a) The right S-module M_S is flat.

(b) ${}_RM$ generates the kernel of every homomorphism ${}_RM^m \rightarrow {}_RM^n$, where m, n are natural numbers. (Here one can also set $n=1$).

Further, R. W. Miller has proved that, in case where ${}_RM$ is finitely generated and projective, the above statements are equivalent to

(c) $\mathscr{C}_2 = \mathscr{C}_3$[3]

Here, regarding to the above results, we shall prove the following

THEOREM. *Let ${}_RM$ be left R-module with the endomorphism ring S, and Q an injective cogenerator in the category ${}_R\mathfrak{M}$ of all left R-modules. Then the following statements are equivalent:*

(1) *M_S is flat.*

(2) *The left S-module ${}_S\mathrm{Hom}_R(M, Q)$ is injective.*

(3) *${}_S\mathrm{Hom}_R(M, Q)$ is absolutely pure, that is, every homomorphism of a finitely generated submodule of ${}_SS^m$ to ${}_S\mathrm{Hom}_R(M, Q)$ is extended to that of ${}_SS^m$.*

(4) *${}_S\mathrm{Hom}_R(M, Q)$ is semi S-injective, that is, every homomorphism of a finitely generated left ideal of S to ${}_S\mathrm{Hom}_R(M, Q)$ is extended*

1) In what follows, we assume that every ring has an identity element and every module is unital.

2) Cf. [5], 1.14 Satz.

3) Cf. [2], Theorem 2.1*.