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A theorem on group spaces

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§ 1. Introduction.

The papers [5] and [9] contain the classification of all finite doubly transitive permutation groups (Ω, G) which have the property that the stabiliser G_{α} on a point $\alpha\in\Omega$ contains a normal subgroup Q acting sharply transitively on $\Omega-\{\alpha\}$. This result has found many applications, especially in finite geometry. However in most of these applications the algebraic objects most immediately connected to the geometric situation are group spaces rather than permutation groups. Therefore it seems advisable to adapt the above result to the more general case of group spaces. To do this is the main purpose of this paper. Also, we investigate, which elements of a group G of the type considered here are the product of two elements lying in conjugates of Q (see Lemma 2.6). Furthermore, in § 3 we present an application of our result to collineation groups of projective planes. Our notation is the same as in [3] and [5]. The following lemma which shall be needed for the proof of our main theorem, may be of independent interest. Under the additional assumption that X is a 2-group or X is a p-group and A is sharply transitive on X/Y it has been proved in [7, Lemma 3.3] and [8, Lemma 2.2 and Lemma 2.3].

LEMMA 1.1. *Let X be a finite group, Y a normal subgroup of X of index at least 3 and A a group of automorphisms of X such that A centralises Y and acts transitively on $X/Y-\{1\}$. Clearly X/Y is an elementary abelian p-group for some prime p. Denote $|X/Y|=p^{n}$ and $C=[X, A]$. Then*

a) *C is a p-group, $X=YC$, $[Y, C]=1$, $C'\leq\Phi(C)\leq C\cap Y\leq\mathfrak{z}C$, $\mathfrak{z}C=C\cap Y$ or $\mathfrak{z}C=C$, and C' is elementary abelian of order at most p^{n-1};*

b) *if $N\leq C$ and N is invariant under A, then $N\leq C\cap Y$ or $N=C$;*

c) *if in addition $p\nmid|A|$, then $C'=\Phi(C)=C\cap Y$, A is faithfully represented on $C/C\cap Y$, and one of the following statements holds:*

I) $X=Y\times C$.

II) *C is a quaternion group of order* 8, $|A|=3$.

III) *$|X/Y|=p^{2}$, where $p\in\{11, 19, 29, 59\}$, C is an extraspecial group of order p^{3}, $C\cap Y=\mathfrak{z}C$ and $A^{(\infty)}\cong SL(2, 5)$.*