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Borel sets in non-separable Banach spaces*)

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1. INTRODUCTION Let E be a Banach space. I shall write $\mathscr{B}=\mathscr{B}(E)$ for the σ-algebra of norm-Borel sets of E and $\mathscr{C}=\mathscr{C}(E)$ for the "cylindrical σ-algebra" of E *i.e.* the smallest σ-algebra of subsets of E for which every element of the continuous dual E' of E is measurable. (By [2], Theorem 2.3 this is just the Baire σ-algebra for the weak topology $\mathscr{T}_s(E, E')$.) Of course $\mathscr{C}\subseteq\mathscr{B}$; if E is separable, $\mathscr{C}=\mathscr{B}$. The question I wish to address in this note is: if $\mathscr{C}=\mathscr{B}$, does it follow that E is separable? This question was suggested to me by S. Okada. I shall show here that the answer is "no".

2. DEFINITIONS (a) If E is a Banach space, the *density character* of E, $d(E)$, is the smallest cardinal of any dense subset of E. Note that if E is infinite-dimensional, then $d(E)$ is also the smallest algebraic dimension of any dense linear subspace of E — the "topological dimension" of E.

(b) If X and Y are any sets and Σ, T are σ-algebras of subsets of X and Y respectively, I shall write $\Sigma\otimes_\sigma T$ for the σ-algebra of subsets of $X\times Y$ generated by $\{A\times B: A\in\Sigma, B\in T\}$. I shall write $\mathscr{P}X$ for the power set of X.

(c) If X is any set. then $\ell^1(X)$ is the Banach space of all functions $x: X\to\boldsymbol{R}$ such that $\|x\|=\Sigma_{t\in X}|x(t)|<\infty$.

3. LEMMA *If X is an infinite set such that $\mathscr{P}(X\times X)=\mathscr{P}X\otimes_\sigma\mathscr{P}X$, then $\mathscr{B}(\ell^1(X))=\mathscr{C}(\ell^1(X))$.*

PROOF (a) I begin with two set-theoretic remarks. First, if Y is any set of the same cardinal as X, then $\mathscr{P}(X\times Y)=\mathscr{P}X\otimes_\sigma\mathscr{P}Y$. Consequently we can use induction to see that, for any $n\geqslant 0$, $\mathscr{P}(X^{n+1})$ is the σ-algebra of subsets of X^{n+1} generated by $\mathscr{R}_n=\{A_0\times\cdots\times A_n: A_i\subseteq X\,\forall\, i\leq n\}$. Secondly the diagonal $\{(t,t): t\in X\}$ belongs to $\mathscr{P}X\otimes_\sigma\mathscr{P}X$, and therefore belongs to the σ-algebra of subsets of $X\times X$ generated by some sequence $\langle A_n\times B_n\rangle_{n\in N}$. Let $\mathscr{E}$ be the countable subalgebra of $\mathscr{P}X$ generated by $\{A_n: n\in\boldsymbol{N}\}\cup\{B_n:$

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