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Cut loci of Berger's spheres

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1. Introduction. In the present note we determine the structure of the cut loci of Berger's spheres explicitly. Let M be a compact riemannian manifold with a fixed point o. Then for every geodesic γ_X (parametrized by arc-length) from o with unit initial direction $X \in T_o M$, we define the cut point $\gamma_X(t)$ of o along γ_X as the last point on γ_X to which the geodesic arc of γ_X minimizes the distance from o. $tX \in T_o M$ will be called the tangent cut point of o along γ_X. The set of all (tangent) cut points of o is called the (tangent) cut loculs of o. The cut locus contains the essential information on the topology of M. Now the structure of cut locus is interesting in connection with the singularlity of the exponential mapping Exp: $T_o M \to M$. Recently in case of analytic riemannian structure or in generic case much progress has been made ([2], [3], [4], [14]). But since their works appeal to the powerful general theory (Hironaka's or Mather's theory), it is not clear how to apply these methods to concrete cases. On the other hand for compact symmetric spaces the structure of cut loci has been completely analyzed in terms of root system by the author and M. Takeuchi ([10], [11], [12]). Thus it is a natural problem to study the cut loci of more general homogeneous riemannian manifolds. But it seems difficult to establish a general theory which analyzes the detailed and concrete structure of cut loci in all homogeneous spaces. So we consider here some examples which seems to be the first step for the above problem. Namely we consider Berger's spheres M_α $(0<\alpha\leq\pi/2)$ —for the construction see § 2— which are homogeneous spaces diffeomorphic to the three dimensional sphere. These riemannian structures may be obtained from the canonical riemannian structure $M_{\pi/2}$ of constant curvature by deforming the metric along the fibers of the Hopf fibering $S^3 \to S^2$ (see [9]) or may be realized as the distance spheres in CP^2 with Fubini-Study metric ([15]) and give nice examples in riemannian geometry ([5], [13], [16]).

Now in the present note, by determining the structure of the cut locus of a point in M_α explicitly we see that the cut locus is the 2-disc whose boundary consists of the conjugate points of order 1 and the cut locus contracts to a point as α converges to $\pi/2$.