Hokkaido Mathematical Journal Vol. 11 (1982) p. 35-38

Some remarks on separable extensions

Dedicated to Professor Goro Azumaya
on his 60th birthday

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(Received May 12, 1980)

Many equivalent conditions for a ring extension to be H-separable have been studied by Sugano and Nakamoto in [2] and [4] etc.. In this connection we give some equivalent conditions for separable extensions in Theorem 1. In § 2, we consider the automorphism group of an H-separable extension, and in Theorem 3 we give an exact sequence which turns into the result of Rosenberg and Zelinsky [3] Theorem 7 in case of Λ is an Azumaya algebra.

§ 1. Separable extension

Throughout this paper we assume that all rings have the identity 1 and subrings contain this element and modules are unitary. For any two-sided module M over a ring A, M^A means the set $\{m\in M| am=ma$ for all $a\in A\}$. Thus for a ring Λ and a subring Γ of Λ, denote by Δ, $\Delta=\Lambda^\Gamma=\{d\in\Lambda| d\gamma=\gamma d,\ \gamma\in\Gamma\}$, the commutator of Γ in Λ, and by C, $C=\Lambda^\Lambda$, the center of Λ.

THEOREM 1. *Let Λ be a ring with the center C, Γ a subring of Λ. Then the following conditions are equivalent.*

(1) *Λ is separable over Γ.*

(2) *$(\Lambda\otimes_\Gamma\Lambda)^\Lambda$ is $(\Lambda\otimes_\Gamma\Lambda)^\Gamma$-finitely generated projective and $\Lambda\cong\mathrm{Hom}_{(\Lambda\otimes_\Gamma\Lambda)^\Gamma}((\Lambda\otimes_\Gamma\Lambda)^\Lambda, \Lambda\otimes_\Gamma\Lambda)$ as two-sided Λ-modules.*

(3) *For any two-sided Λ-module M, $M^\Lambda\cong(\Lambda\otimes_\Gamma\Lambda)^\Lambda\otimes_{(\Lambda\otimes_\Gamma\Lambda)^\Gamma} M^\Gamma$ as C-modules.*

(4) *$C\cong(\Lambda\otimes_\Gamma\Lambda)^\Lambda\otimes_{(\Lambda\otimes_\Gamma\Lambda)^\Gamma}\Delta$ as C-modules.*

(5) *$(\Lambda\otimes_\Gamma\Lambda)^\Lambda\cdot\Delta=C$.*

(6) *There exists an element $\sum x_i\otimes y_i$ in $(\Lambda\otimes_\Gamma\Lambda)^\Lambda$ such that $\sum x_iy_i=1$.*

PROOF. (1)⇒(2). By the definition Λ is separable over Γ means that Λ is a two-sided Λ-direct summand of $\Lambda\otimes_\Gamma\Lambda$, ${}_\Lambda\Lambda_\Lambda<\oplus\Lambda\otimes_\Gamma\Lambda$. Then by Theorem 1.2 [1], (1) implies (2). Note that, in this case, $(\Lambda\otimes_\Gamma\Lambda)^\Lambda$ is a direct summand right ideal of $(\Lambda\otimes_\Gamma\Lambda)^\Gamma\cong\mathrm{Hom}_{\Lambda,\Lambda}(\Lambda\otimes_\Gamma\Lambda, \Lambda\otimes_\Gamma\Lambda)$. (2)⇒(3).