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Invariant flat projective structures on homogeneous spaces

By Yoshio AGAOKA

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Introduction

Let M be an n-dimensional $(n\geqq 2)$ manifold with a projective structure. It is well known that there exists a unique projective normal Cartan connection which induces the original projective structure. E. Cartan proved this fact locally by his method of moving frames ([1], [2]), and later it was settled in the rigorous form using principal fibre bundles by Tanaka, Kobayashi and Nagano ([5], [6], [11]).

In this paper we shall study, as an application of the theory of Cartan connections, invariant flat projective structures (which we abbreviate IFPS) on a homogeneous space $M=L/K$. Our first main result is that there exists a natural one-to-one correspondence between the set of IFPS on $M=L/K$ and the set of projective equivalence classes of Lie algebra homomorphisms $f:\mathfrak{l}\rightarrow\mathfrak{sl}(n+1, \boldsymbol{R})$ ($\mathfrak{l}$ is a Lie algebra of L) satisfying certain conditions (Theorem 2. 12). This is a natural generalization of the classical theory concerning invariant affine connections on $M=L/K$ (cf. Vol. II [7]). Using this correspondence we can determine the existence or non-existence of IFPS on many real simple Lie groups and irreducible Riemannian symmetric spaces. It will be proved that, among them, $M=SO(3)$, $SL(m, \boldsymbol{R})$ $(m\geqq 2)$, $SU^{*}(2m)$ $(m\geqq 2)$, $SL(m, \boldsymbol{R})/SO(m)$ $(m\geqq 2)$ and $SU^{*}(2m)/Sp(m)$ $(m\geqq 2)$ admit an IFPS. We determine the number of IFPS on these spaces (Theorem 5. 3, 5. 11, 7. 1 and 8. 5). For example $M=SL(m, \boldsymbol{R})/SO(m)$ $(m\geqq 3)$ admits two projectively flat invariant affine connections and neither of them is the canonical (Riemannian) connection.

In [5] and [6] a projective structure on M is defined by a reduction of structure group of $P^{2}(M)$, the bundle of 2-jet frames over M, to a certain subgroup of $G^{2}(n)$. But for the later convenience we take the standpoint of Tanaka [12], not using the jet theory. For this reason, in § 1 we review the theory of projective Cartan connections and fix our notations, following [12]. In § 2 we prove the first main result in this paper (Theorem 2. 12). Let $M=L/K$ be an n-dimensional homogeneous space with an invariant