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Regular sequences of ideals in a noncommutative ring

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ABSTRACT: Let R be an associative ring which is not in general commutative. If M is a left R-module we define the notion of an M-regular ideal of R and the notion of an M-regular sequence of ideals of R, generalizing the corresponding notions in commutative ring theory. If M is finitely-generated and if R is left stable and left noetherian, a bound is given for the lengths of M-regular sequences of ideals.

0. Background and notation. Throughout the following, R will denote an associative (but not necessarily commutative) ring with unit element 1. The word "ideal" will mean "proper two-sided ideal" unless modified by an adjective indicating dexterity. The complete brouwerian lattice of all (hereditary) torsion theories defined on the category R-mod of unitary left R-modules will be denoted by R-tors. Notation and terminology regarding such theories will follow [2]. In particular, if M is a left R-module then $\xi(M)$ will denote the smallest torsion theory relative to which M is torsion and $\chi(M)$ will denote the largest torsion theory relative to which M is torsionfree. The unique minimal element of R-tors is $\xi=\xi(0)$ and the unique maximal element of R-tors is $\chi=\chi(0)$. A torsion theory in R-tors is said to be *stable* if and only if its class of torsion modules is closed under taking injective hulls. The ring R is said to be *left stable* if and only if every element of R-tors is stable.

If $\tau\in R$-tors then a nonzero left R-module M is called *τ-cocritical* if and only if it is τ-torsionfree while each of its proper homomorphic images is τ-torsion. A left R-module M is *cocritical* if and only if it is $\chi(M)$-cocritical. A torsion theory in R-tors is said to be *prime* if and only if it is of the form $\chi(M)$ for some cocritical left R-module M. If M is a nonzero left R-module then $\{\chi(N)|N$ is a cocritical submodule of $M\}$ is called the set of *associated primes* of M and is denoted by $\mathrm{ass}(M)$.

1. Regular ideals and elements. Let M be a nonzero left R-module. We will say that an ideal I of the ring R is *M-regular* if and only if M is $\xi(R/I)$-torsionfree. An element a of R will be said to be *M-regular* if and only if the ideal RaR generated by a is M-regular. The following is an elementary, and essentially well-known, characterization of regular ideals.