Hokkaido Mathematical Journal Vol. 12 (1983) p. 1-9

Principal functions and invariant subspaces of hyponormal operators

By Yoshihiro NAKAMURA

(Received March 24, 1982)

1. Introduction and Theorems

A bounded linear operator T on a Hilbert space is said to be *hyponormal* if its self-commutator $[T^{*}, T]\equiv T^{*}T-TT^{*}$ is positive semi-definite, and *pure hyponormal* if, in addition, T has no nontrivial reducing subspace on which it is normal.

It is not known at present whether every hyponormal operator has a non-trivial invariant subspace. Putnam [7] and Apostol and Clancey [1] presented some conditions for a hyponormal operator to have invariant subspaces. In this paper, by using the principal function invented by Pincus [6], we shall improve the results of Putnam, and Apostol and Clancey.

Let $T=X+iY$ be a pure hyponormal operator, where X and Y are self-adjoint. Then it is known that X and Y are absolutely continuous (see [4, Chap. 2, Th. 3. 2]). Let $X=\int tdG(t)$ be the spectral resolution of X. Then the *absolutely continuous support* E_{X} of X is defined as a Borel subset of the real line (determined uniquely up to a null set) having the least Lebesgue measure and satisfying $G(E_{X})=I$. Analogously E_{Y} is defined for Y.

The main results in this paper are the following:

THEOREM 1. *Let $T=X+iY$ be a pure hyponormal operator. Suppose that, for some real μ_{0}, the spectrum of T, $\sigma(T)$, has non-empty intersection with each of the open half-planes $\{z: \mathrm{Re}\, z<\mu_{0}\}$ and $\{z: \mathrm{Re}\, z>\mu_{0}\}$. If*

$$\int_{E_{X}}\frac{F(x)}{(x-\mu_{0})^{2}}dx<\infty$$

where $F(x)$ is the linear measure of the vertical cross section $\sigma(T)\cap\{z: \mathrm{Re}\, z=x\}$, then T has a non-trivial invariant subspace.

THEOREM 2. *In Theorem 1 the existence of a non-trivial invariant subspace is also guaranteed if the integrability condition is replaced by*

$$\int_{E_{X}}\frac{1}{|x-\mu_{0}|}dx<\infty .$$