Hokkaido Mathematical Journal Vol. 12 (1983) p. 10-16

Projective Γ-sets

By Tomoyuki YOSHIDA

(Received January 26, 1982)

The purpose of this paper is to study projective functors from a small category to the category of sets. Our results are generalizations of semigroup cases.

1. Some basic definitions and properties

Let $\mathscr{S}$ be the category of sets and Γ a small category. We denote by $\mathscr{S}^{\Gamma}$ the functor category from Γ to $\mathscr{S}$. Then an object in $\mathscr{S}^{\Gamma}$ is called a (*right*) *Γ-set* and a morphism in $\mathscr{S}^{\Gamma}$ is called a *Γ-map*. We denote the hom-set from i to j in Γ by $\Gamma(i, j)$. Furthermore *throughout the paper*, we denote the composition $X \xrightarrow{f} Y \xrightarrow{g} Z$ by $fg: X \to Z$. So a (right) Γ-set X consists of sets X_i, $i \in \Gamma$, which are called *stalks* at i, together with maps

$$X_i \times \Gamma(i, j) \longrightarrow X_j : (x_i, \alpha) \longmapsto x_i \cdot \alpha$$

for $i, j \in \Gamma$ which satisfy the conditions:
(a) $x_i \cdot 1_i = x_i$ for $i \in \Gamma$, $x_i \in X_i$ and the identity 1_i;
(b) $(x_i \cdot \alpha) \cdot \beta = x_i \cdot (\alpha\beta)$ for $x_i \in X_i$, $\alpha \in \Gamma(i, j)$, $\beta \in \Gamma(j, k)$.
Furthermore a Γ-map $f: X \to Y$ between Γ-sets is a family of maps f_i: $X_i \to Y_i$, $i \in \Gamma$, satisfying the condition $(x_i \cdot \alpha) f_j = (x_i f_i) \cdot \alpha$ for $i, j \in \Gamma$, $x_i \in X_i$, $\alpha \in \Gamma(i, j)$. The set of all Γ-maps of X to Y is denoted by $\Gamma(X, Y)$. We define analogously *left* Γ-sets which can be regarded as contravariant functor from Γ to $\mathscr{S}$. The category of Γ-sets, $\mathscr{S}^{\Gamma}$, is complete and cocomplete. In fact, limits and colimits of Γ-sets are constructed pointwise. A Γ-set X is called a *finite* Γ-set provided each stalk X_i is a finite set. The full subcategory of finite Γ-sets in $\mathscr{S}^{\Gamma}$ is denoted by $\mathscr{S}^{\Gamma}_f$.

Any set A is regarded as a *constant* Γ-set defined by $A_i = A$ and $a \cdot \alpha = a$ for all $a \in A_i$, $\alpha \in \Gamma(i, j)$. For each $k \in \Gamma$, the hom-functor $H^k: i \mapsto \Gamma(k, i)$ is a Γ-set, which is called a *representable* Γ-set. Of course the map $H^k{}_i \times \Gamma(i, j) \to H^k{}_j : (\gamma, \alpha) \mapsto \gamma\alpha$ is defined by the compositions. If Γ is a finite category, that is, all morphisms in Γ makes a finite set, then H^k is a finite Γ-set. It is well-known as the Yoneda Lemma that Γ-maps of H^k to X are bijectively corresponding with elements of X_k. The Yoneda embedding Y: $\Gamma^{op} \to \mathscr{S}^{\Gamma} : k \mapsto H^k$ is fully-faithful, and furthermore Y preserves and reflects