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On infinitesimal $C_{2\pi}$-deformations of standard metrics on spheres

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Introduction

Let M be a riemannian manifold and g its riemannian metric. Then we call M a C_l-manifold and g a C_l-metric if all of its geodesics are closed and have the common length l. As is well-known, the unit sphere S^n in the euclidian space $\boldsymbol{R}^{n+1}$ equipeed with the induced metric (the standard metric) g_0 is a $C_{2\pi}$-manifold.

Let us consider a one-parameter family $\{g_t\}$ of $C_{2\pi}$-metrics on S^n such that $g_0=g_t|_{t=0}$ is the standard one. Put

$$h=\frac{d}{dt}g_t|_{t=0}\,.$$

We shall call such a family $\{g_t\}$ a $C_{2\pi}$-deformation of the standard metric g_0, and h an infinitesimal $C_{2\pi}$-deformation of g_0. It is known that each infinitesimal $C_{2\pi}$-deformation h satisfies the so-called zero-energy condition, i. e.,

$$\int_0^{2\pi} h\left(\dot{\gamma}(s), \dot{\gamma}(s)\right) ds=0$$

for any geodesic $\gamma(s)$ of (S^n, g_0) parametrized by arc-length (cf. [1] p. 151). We denote by $\mathscr{K}^2$ the vector space of symmetric 2-forms on S^n which satisfy the zero-energy condition.

In his paper [3] Guillemin proved that in the case of S^2 any symmetric 2-form $h\in\mathscr{K}^2$ is necessarily an infinitesimal $C_{2\pi}$-deformation of g_0. On the other hand, for $C_{2\pi}$-deformations on S^n $(n\geq 3)$, the examples constructed by Weinstein ([1] p. 119) are all that we know up to now, and the corresponding infinitesimal $C_{2\pi}$-deformations form a rather small subset of $\mathscr{K}^2$.

The main purpose of this paper is to introduce and study a necessary condition for a symmetric 2-form $h\in\mathscr{K}^2$ to be an infinitesimal $C_{2\pi}$-deformation of the standard metric g_0 on the n-dimensional sphere S^n $(n\geq 3)$. This condition is called the second order condition, and is naturally obtained through the interpretation of the $C_{2\pi}$-property in terms of the symplectic geometry on the cotangent bundle T^*S^n (Proposition 1. 4).