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# A remark on socles and normal subgroups

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## 1. Introduction

In modular representation theory of finite groups, it is often usefull to study socles of indecomposable modules. For example, the famous Brauer's result on socles and heads of projective indecomposable modules, or Green's recent result [5] on socles and heads of indecomposable direct summands of some permutation modules. In this paper, we are concerned with socles of direct summand of modules indeuced from normal subgroups.

Let $G$ be a finite group, $p$ a rational prime, and $k$ a field of characteristic $p$. Let $K$ be a normal subgroup of $G$ and $W$ an indecomposable $kK$-module. A number of authoers, including A. H. Clifford, have investigated the induced module $W^G$ and the endomorphism ring $\mathrm{End}_{kG}(W^G)$. In his paper [1], S. B. Conlon proved $\mathrm{End}_{kG}(W^G)$ is almost isomorphic to a twisted group algebra over $G/K$. After, P. A. Tucker [7], [8] and H. N. Ward [9] studied the relationship between submodules of $W^G$ and left ideals of $\mathrm{End}_{kG}(W^G)$ in case $W$ is a simple $kK$-module. Their results are found in the book [2]. As corollary of them, we see easily, if $W$ is a simple $kK$-module and the innertia group $T_G(W)$ equals $G$, then any indecomposable direct summand of $W^G$ has a simple socle and a simple head. We shall extend it and prove the following Theorem :

THEOREM 3.2 *Let $k$ be an algebraically closed field. Suppose $K$ is a normal subgroup of $G$ and $W$ is an indecomposable $kK$-module satisfying $T_G(W)=G$. If the socle* $\mathrm{soc}(W)$ *is a simple $kK$-module, then for an indecomposable direct summand $V$ of $W^G$ the socle* $\mathrm{soc}(V)$ *is a simple $kG$-module.*

*Notation.* Maps are usually on the left with the corresponding conversion for writing compositions. Let $A$ and $B$ sets and $f$ a map of $A$ to $B$. For a subset $C$ of $A$ we denote by $f|_C$ the restriction of $f$ to $C$. For a subgroup $H$ of $G$ we denote by $(G/H)$ a set of representatives of the left coset $gH$ in $G$, containing the identity element. All $kG$-modules are finite generated left $kG$-modules. For a ring $A$ we denote by $J(A)$ the Jacobson radical of $A$. Let $M$ and $M'$ be $kG$-modules. The socle of $M$ is the maxmal