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Transformation equations and the special values of Shimura's zeta functions

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Introduction

As is well known, the study of the transformation equations for modular forms has one of its origins in Klein's work [5], and many authors, e.g., Kiepert [4], Hurwitz [3], Fricke [1] and Herglotz [2], did certain contributions in this theory. For a modular form h of weight k on the congruence subgroup $\Gamma_0(N)$ of $SL_2(\mathbf{Z})$, the transformation equation for h is defined by

$$\Phi(X;\ h)=\prod_{\alpha\in\Gamma_0(N)\backslash SL_2(\mathbf{Z})}(X-h|_k\alpha)=0\,,$$

where $h|_k\alpha$ denotes the usual action of $\alpha\in SL_2(\mathbf{Z})$ of weight k (for the notation, see § 1). The above mentioned references are mainly concerned with $\Delta(Nz)$ and related functions as h for the discriminant function Δ.

For a long time, the importance of the investigation of these equations has been in the mind of the first author of this paper.

Now recently, Shimura [11] proved the algebraicity at certain integers of the zeta function defined by

$$D(s,f,g)=\sum_{n=1}^{\infty}a(n)\,b(n)\,n^{-s}\,,$$

where $f=\sum_{n=1}^{\infty}a(n)\,e(nz)$ $(e(z)=\exp(2\pi iz))$ is a primitive cusp form on $\Gamma_0(N)$ of weight k and $g=\sum_{n=0}^{\infty}b(n)\,e(nz)$ is an arithmetic modular form on $\Gamma_0(N)$ of weight l less than k. Then $D(m,f,g)$ for $(k+l)/2-1<m<k$ is an algebraic number times the Petersson self inner product of f and a power of π. We take as h the product of g and a certain Eisenstein series $E_{\lambda,N}^{*}$ which is utilized in his proof of the algebraicity of $D(m,f,g)$. Then the sum of μ-th power of all the roots of $\Phi(X;\ h)=0$ can be expressed as a finite linear combination of primitive forms of level 1 with the coefficients $D(m,f,g')$ for $g'=g^{\mu}(E_{\lambda,N}^{*})^{\mu-1}$. In fact, we have

THEOREM. *For an arbitrary element $g\in S_l(\Gamma_0(N))$ and for any positive integers μ and $\lambda>2$, we have*

(i) $$\mathrm{Tr}\,(gE_{\lambda,N}^{*})^{\mu}=c\sum_{f\in P(k\mu)}\frac{D(k\mu-1,f,g^{\mu}(E_{\lambda,N}^{*})^{\mu-1})}{\pi^{k\mu}\langle f,f\rangle}f$$