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Convex programming on spaces of measurable functions

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1. Introduction. In [3] and [4], J. Zowe has considered convex programming with values in ordered vector spaces. His hypotheses are so restrictive that his theory does not apply to the case of the function spaces L_p $(\infty>p>0)$. For L_∞ and $C(X)$ with X a Stonean space, his method is very useful. We shall consider in this note convex operators whose values are in much more general spaces than the usual function spaces such as L_p. Functions assuming the value $+\infty$ introduce certain complications, to which we must address ourselves.

For simplicity, we consider only convex operators defined on the real line $\mathbb{R}$ with values in the space of measurable functions with values in $\mathbb{R}\cup\{+\infty\}$. The general case will be treated in a later publication.

In this note, we present a generalization of the Fenchel-Moreau theorem and also of the Fenchel theorem. It is appropriate to consider the $P(\Omega)$ of measurable functions, whose definition will be found in Section 2.

2. Preliminary lemmas

Let F be an extended real-valued function on the real numbers $\mathbb{R}$, possibly assuming the value $+\infty$, but not the value $-\infty$. Let D be a (dense) countable subfield of $\mathbb{R}$. Such an extended real-valued function F defined on D is said to be *D-convex* if

$$F(\alpha x+\beta y)\leqq\alpha F(x)+\beta F(y)$$

for $\alpha,\beta\in D$ with $\alpha+\beta=1$, $\alpha,\beta\geq 0$ and $x, y\in D$.

An $\mathbb{R}$-convex function will be called convex, as usual.

We first present a number of lemmas.

LEMMA 1. *Every finite-valued D-convex function defined on D is continuous in D: that is $x_n\rightarrow x(x_n, x\in D)$ implies that $F(x_n)\rightarrow F(x)$.*

PROOF. If the sequence $F(x_n)$ does not converge for $x_n\rightarrow x$, the convexity of F implies that either $F(y)=+\infty$ for all $y>x$ or $F(y)=+\infty$ for all $y<x$. Since F is finite-valued, F is continuous.

Curiously enough, a D-convex function F defined on all of $\mathbb{R}$, i. e. a function satisfying $F(\alpha x+\beta y)\leqq\alpha F(x)+\beta F(y)$ for $\alpha,\beta\in D$ with $\alpha+\beta=1$, $\alpha,\beta\geq 0$ and $x, y\in\mathbb{R}$, is not necessarily continuous on $\mathbb{R}$. (For example, a