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Automorphism groups of Σ_{n+1}-invariant trilinear forms

By Yoshimi EGAWA and Hiroshi SUZUKI

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1. Introduction

Let Σ_{n+1} be the symmetric group on the set $\{0, 1, \cdots, n\}$ of cardinality $n+1$, $n \geqq 2$. Let $V=\langle e_1, \cdots, e_n\rangle$ be a natural n-dimensional irreducible Σ_{n+1}-module over the complex number field $\mathbf{C}$. (That is, $\{e_1, \cdots, e_n\}$ is a basis of V such that if we let $e_0=-(e_1+\cdots+e_n)$, then Σ_{n+1} acts on $\{e_0, e_1, \cdots, e_n\}$ in the standard way.) We regard Σ_{n+1} as a subgroup of $GL(V)$. We define a Σ_{n+1}-invariant symmetric trilinear form θ_n on V by

$\theta_n(e_j, e_j, e_j)=n(n-1)$, $1 \leqq j \leqq n$;

$\theta_n(e_j, e_j, e_k)=-(n-1)$, $1 \leqq j, k \leqq n$, $j \neq k$;

$\theta_n(e_j, e_k, e_h)=2$, $1 \leqq j, k, h \leqq n$, $j \neq k \neq h \neq j$.

Now we can state our main results.

THEOREM 1. *Let Σ_{n+1}, V, θ_n be as above. Let θ be an arbitrary nonzero Σ_{n+1}-invariant symmetric trilinear form on V. Then*

$\theta=\alpha\theta_n$, $0 \neq \alpha \in \mathbf{C}$

and so $Aut\theta=Aut\theta_n$, where we define the automorphism group of θ to be

$Aut\theta=\{\sigma \in GL(V): \theta(x^\sigma, y^\sigma, z^\sigma)=\theta(x, y, z)$ *for all* $x, y, z \in V\}$.

THEOREM 2. *If $n=2$ or $n \geqq 4$,*

$Aut\theta_n=\langle \omega I\rangle \times \Sigma_{n+1}$,

where I is the identity element of $GL(V)$ and $\omega=(-1+\sqrt{3}i)/2$.

REMARK. The structure of $Aut\theta_3$ is described in LEMMA 2. 3.

If n is odd, our proof of THEOREM 2 is essentially an elementary analysis of the action of $Aut\theta_n$ on the set of "singular" elements of V. If n is even, we first prove that there is no singular element, which implies that $Aut\theta_n$ is finite by [6, THEOREM B]. We then apply a deep result of H. Bender [3] to complete the proof.

Symmetric bilinear and trilinear mappings

$V \times V \longrightarrow V$, $V \times V \times V \longrightarrow V$,

which are Σ_{n+1}-invariant are studied by K. HARADA [5] and by the second author [7], respectively. Our result here is analogous to that of the bilinear mapping case. This is natural, because

$V \times V \times V \longrightarrow \mathbf{C}$