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On the curvature of Riemannian submanifolds of codimension 2

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Introduction

Let (M, g) be an n-dimensional Riemannian manifold which is isometrically immersed into the $n+k$-dimensional Euclidean space $\boldsymbol{R}^{n+k}$. Then the curvature transformation R of (M, g) satisfies the condition

$$(*) \qquad \operatorname{rank} R(X, Y) \leqq 2k$$

for any tangent vectors $X, Y \in T_xM$, where we consider $R(X, Y)$ as a linear endomorphism of T_xM. Using this condition, Agaoka and Kaneda gave in [4] some estimates on the dimension of the Euclidean space into which Riemannian symmetric spaces can be locally isometrically immersed. For example they proved that the complex projective space $P^n(\boldsymbol{C})$ cannot be locally isometrically immersed in codimension $n-1$. But if $k \geqq (n-1)/2$, the condition $(*)$ does not impose any restrictions on the curvature of n-dimensional Riemannian submanifolds of $\boldsymbol{R}^{n+k}$.

Our first purpose of this paper is, using the representation theory of $GL(n, \boldsymbol{R})$, to determine the polynomial relations of the curvature tensor of $M^n \subset \boldsymbol{R}^{n+k}$, up to degree 3 explicitly (Theorem 1.4) and to find a new condition on the curvature tensor which serves as the obstruction in the cases $M^4 \subset \boldsymbol{R}^6$ and $M^5 \subset \boldsymbol{R}^7$. (See §1 and §2. Note that in these cases, the inequality $(*)$ reduces to a trivial condition.) Our second purpose is to express this new relation appeared in degree 3 in a simple form which is easy to calculate (Proposition 3.3, Theorem 3.4). As applications of this curvature relation, we prove that Riemannian symmetric spaces $P^2(\boldsymbol{C})$, $SU(3)/SO(3)$ and their non-compact dual spaces cannot be isometrically immersed in codimension 2 even locally (Corollary 3.5). As for $P^2(\boldsymbol{C})$ and its dual space, this result can be proved, using the theorems in Ôtsuki [18] and Weinstein [23] (see Remark (1) after Corollary 3.5). But, as for $SU(3)/SO(3)$ and its dual space, this is a new result, which cannot be obtained by a previously known method.

Now we explain our method briefly. Let V be an n-dimensional real vector space and let K be the space of curvature like tensors on V (see §1). We define a quadratic map $\gamma_k : S^2 V^* \otimes \boldsymbol{R}^k \longrightarrow K$ by