Hokkaido Mathematical Journal Vol. 16 (1987), p. 81-95

THE F. AND M. RIESZ THEOREM AND SOME FUNCTION SPACES

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(Received March 12, 1986, Revised August 11, 1986)

§1. Introduction

In 1916 F. and M. Riesz published the following result: if μ is a bounded complex Borel measure on the unit circle $\boldsymbol{T}$ such that

$$\hat{\mu}(n)=\int_{\boldsymbol{T}}e^{-in\theta}d\mu(\theta)=0 \qquad \text{for } n=-1,\ -2,\ldots,$$

then μ is absolutely continuous with respect to the Lebesgue measure on $\boldsymbol{T}$. Some forty years later, Helson and Lowdenslager generalized this theorem to compact Abelian groups with ordered dual ([5]). Since then a number of related results have been obtained under more general settings ([1], [2], [4], [6], [10], [15], [17]).

In his papers [13] and [14] Sarason showed that $H^{\infty}(\boldsymbol{T})+C(\boldsymbol{T})$ is a closed subalgebra of $L^{\infty}(\boldsymbol{T})$, and that $H^{\infty}(\boldsymbol{R})+C_{u}(\boldsymbol{R})$ is a closed subalgebra of $L^{\infty}(\boldsymbol{R})$. Subsequently, Rudin [12] and Yamaguchi [16] investigated spaces of type $H^{\infty}+C$ on locally compact Abelian groups with ordered dual.

Meanwhile, Hewitt, Koshi, and the author recently presented simple proofs for results in [1], [2], and [17] and recognized that the embedding theorem of a locally compact Abelian group into a locally compact divisible Abelian group is useful in dealing with more general subsemigroups instead of orders ([7]). In the present paper we continue to use the embedding theorem and study the relation between the F. and M. Riesz theorem and spaces of type $H^{\infty}+C$.

In section 2 we describe our notation and state main theorem which gives a generalization of a theorem of Yamaguchi ([16]]). In fact our result supplies more information on the relation between the F. and M. Riesz theorem and spaces of type $H^{\infty}+C$. Some preliminary lemmas are proved in section 3. We give the proof of our main theorem in section 4.

§2. Notation and Main Theorem

Throughout this paper, the symbols $\boldsymbol{Z}$, $\boldsymbol{Z}_{+}$, $\boldsymbol{R}$, $\boldsymbol{R}_{+}$, and $\boldsymbol{T}$ will denote the integers, the nonnegative integers, the real numbers, the nonnegative real numbers, and the circle group respectively and the term " locally compact