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Ergodic H^1 Is Not A Dual Space

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§1. Introduction

Suppose that X is a standard Borel measure space with probability measure m and that $\{T_t\}_{t\in\boldsymbol{R}}$ is an ergodic, measurable action of $\boldsymbol{R}$ on X preserving m. Via composition, $\{T_t\}_{t\in\boldsymbol{R}}$ acts on functions over X. $(T_tf)(x)=f(T_tx)$, and when restricted to $L^p(X)$, $p<\infty$, $\{T_t\}_{t\in\boldsymbol{R}}$ is strongly continuous; on $L^\infty(X)$, $\{T_t\}_{t\in\boldsymbol{R}}$ is only weak-* continuous. Ergodic H^∞, $H^\infty(X)$, is defined to be the subspace of $f\in L^\infty(X)$ such that, for almost all x, the function of t, $f(T_tx)$, lies in $\mathrm{H}^\infty(\boldsymbol{R})$; i.e. this function admits an extension to a bounded analytic function in the upper half-plane. For p in the range $0<p<\infty$, ergodic H^p, $H^p(X)$, is defined to be the closure of $H^\infty(X)$ in $L^p(X)$. As is shown in [10], when $1\leq p$, $H^p(X)$ is the subspace of all $f\in L^p(X)$ such that, with the exception of a null set of x, the function of t, $f(T_tx)$, when divided by $t+i$ lies in the usual Hardy space $H^p(\boldsymbol{R})$ associated with the upper half-plane. Our objective is to prove the theorem that is our title.

THEOREM. *If $\{T_t\}_{t\in\boldsymbol{R}}$ is not periodic, then $H^1(X)$ is not a dual space.*

This result was conjectured by the second author in [11] and it was noted there that the theorem is true if $\{T_t\}_{t\in\boldsymbol{R}}$ has pure point spectrum. In this case X is a quotient of the Bohr groups and harmonic analysis on X is the key to the proof (see [8]). In our more general setting this tool is not at our disposal. Of course. if $\{T_t\}_{t\in\boldsymbol{R}}$ is periodic, then $H^1(X)$ is (isometrically isomorphic to) the classical Hardy space, $H^1(\boldsymbol{T})$, on the circle $\boldsymbol{T}$, and this space is well-known to be a dual space by the F. and M. Riesz theorem. Namely, $H^1(\boldsymbol{T})$ is the dual of the quotient space $C(\boldsymbol{T})/A_0(\boldsymbol{T})$, where $A_0(\boldsymbol{T})$ is the space of (boundary values of) functions which are continuous on the closed unit disc, analytic on the interior, and vanish at the origin. The proof is simple. The dual of $C(\boldsymbol{T})$ is the space of measures on

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