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Perfect Sets and Sets of Multiplicity

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§ **1.** The class of closed, uncountable sets and the class of closed sets of multiplicity (M-sets) have been very extensively investigated; each occurs as the class of removable sets for some problems in analysis. We describe some examples in which the first class is represented, roughly speaking, by the second. Let F be a compact set in Euclidean space and f a continuous map of F onto a Cantor set C. We require that

(a) For each element x of C, $f^{-1}(x)$ is a U-set (that is, not an M-set). The same is then true for $f^{-1}(A)$, whenever A is a closed, countable subset of C.

(b) For each perfect set P in C, $f^{-1}(P)$ is an M-set. We call such functions c. m. mappings (cardinality-multiplicity mappings) simply to have a name for them. The thee examples of c. m. mappings that follow are based on different principles and have specific properties that cannot be attained by a single construction. The idea of representing classes of sets by inverse images is from [3]; further comparisons are postponed to § 5.

§ **2. A simple example in R^2.** In this example and the next one, property (b) occurs in a stronger, somewhat peculiar form:

(b′) For each perfect set P in C, $f^{-1}(P)$ carries a probability measure μ, such that $\mu*\mu$ is absolutely continuous. Thus $f^{-1}(P)$ is an M_0-set.

Let C be represented as a closed set on the arc $0\leq\theta\leq\pi$ of the unit circle, let F be the set $\{re^{i\theta}: \theta\in C, 1\leq r\leq 2\}$ and let $f(re^{i\theta})=\theta$, so that (a) is obvious. As for (b′), let P be a perfect set in C, so that P carries a continuous probability measure λ. Then $f^{-1}(P)$ carries the measure $\mu=\lambda(d\theta)dr$, and assertion (b′) is simply an observation about the convolution of the linear measures on line segments which aren't parallel.

§ **3. An example in R^1.** We begin with an outline. To each element x in C we attach a probability measure $\mu(x)$, whose support, say $F(x)$, is a U-set; moreover $\mu(x)*\mu(y)$ is absolutely continuous whenever $x\neq y$. Then the function f is defined so that $f^{-1}(x)=F(x)$ for each x in C. The proof that f is single-valued and continuous is the most difficult point, and requires a detour in the method.