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Spectral relations and unitary mixing in semifinite von Neumann algebras

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Abstract

Let $\mathscr{M}$ be a semifinite von Neumann algebra on a separable Hilbert space with a faithful normal semifinite trace τ on $\mathscr{M}$. Between τ-measurable operators, the condition of unitary mixing and other similar ones are characterized in terms of the spectral relations such as the (sub)majorization. Among other things, it is proved that, for positive x and y in $L^1(\mathscr{M};\ \tau)$, x is in the $\|\cdot\|_1$-closed convex hull of the unitary orbit of y if and only if the majorization $ex \prec ey$ holds for every central projection e.

Introduction

As a noncommutative measure space, let $(\mathscr{M}, \tau)$ be a pair of a semifinite von Neumann algebra $\mathscr{M}$ and a faithful normal semifinite trace τ on $\mathscr{M}$. The noncommutative integration theory (in the semifinite case) was initiated by Segal [27] and Dixmier [10] (also [24]), and the noncommutative probability theory was developed by Umegaki [32]. The concept of τ-measurable operators was introduced by Nelson [22]. The space $\tilde{\mathscr{M}}$ of τ-measurable operators affiliated with $\mathscr{M}$ gives a nice foundation for the noncommutative L^p-spaces $L^p(\mathscr{M};\ \tau)$. The notion of generalized s-numbers of τ-measurable operators extends the usual s-numbers of compact operators and the decreasing rearrangements of measurable functions. This notion has been studied in some contexts by several authors (see [12, 14, 25, 28, 33] for instance). Recently Fack and Kosaki [13] established an extensive and unified exposition on generalized s-numbers of τ-measurable operators.

Between positive selfadjoint x and y in $\tilde{\mathscr{M}}$, the spectral relations of majorization $x \prec y$, submajorization $x \prec_w y$, spectral dominance $x \lesssim y$ and spectral equivalence $x \approx y$ are defined by means of the generalized s-numbers of x and y. The precise definitions of these will be given in § 1 of this paper. The notions of majorization and submajorization have been extensively studied in theory of matrices (see Marshall and Olkin [21] and Ando [3]). We discussed in [15] those spectral relations in connection with doubly (sub)stochasic maps on $\mathscr{M}$. Furthermore, when $\mathscr{M}$ is a factor, we char-