Hokkaido Mathematical Journal Vol. 18 (1989), p. 231-243

Radon, Baire, and Borel measures on compact spaces. I

H. H. SCHAEFER
(Received August 16, 1988)

Introduction

There are essentially two approaches to the classical theory of integration on locally compact spaces; one is represented by Halmos [5] (see also [3]), the other by Bourbaki [1]. One way to illuminate their mutual relationship is to employ the theorem of Riesz-Markov [3], to the effect that every Radon measure on a compact space X possesses a unique regular Borel extension, and to imbed the space $B(X)$ of bounded Borel functions into the bidual of $C(X)$. While the Riesz-Markov Theorem itself is a deep, not readily proved result, it does not immediately reveal the finer aspects of the connections between continuous, Baire, and Borel functions or between the corresponding types of measures.

The purpose of this treatise is to make these connections and relations as transparent as possible. The approach is Bourbaki's approach turned upside down, as it were, at least in the beginning: it is (or so the author believes) utterly functional analytic, the key notion being that of a Riesz space. The present first part is concerned essentially with imbedding the Riesz spaces of bounded Baire and Borel functions into $C(X)''$, this latter space provided, in addition to its norm topology, with the topology $o(C(X)'', M(X))$ (see below) under which it is a complete Lebesgue space in the sense of Fremlin [4]. The simultaneous discussion of all Radon, Baire, and Borel measures on X is facilitated by restriction to compact spaces X and to Riesz spaces of bounded functions on X. Extension, where appropriate, to unbounded functions and measures is usually easy. Notation and terminology is standard or follows [6], [7].

A. The General Setting

A. 1 *Notation.* Let X denote a compact space. By $C(X)$, $M(X)$, $\bar{C}(X)$ we will understand, respectively, the Banach space of real continuous functions on X, its dual Banach space, and its bidual. These three spaces are Banach lattices under their natural orderings. Moreover, on