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Gap theorems for Riemannian manifolds of constant curvature outside a compact set

Dedicated to Professor Noboru Tanaka on his 60th birthday

by Kunio SUGAHARA

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§ 1. Introduction and Main Results.

In Riemannian geometry, the spaces of constant curvature are considered to be the typical models in various problems. For example, the famous sphere theorem deals with compact Riemannian manifolds whose sectional curvatures are similar to that of the standard sphere. For noncompact Riemannian manifolds, we have the following theorem. Let K_M denote the sectional curvature of a Riemannian manifold M and k be a nonpositive constant.

GAP THEOREM 1.1 (Greene-Wu[5], Kasue-Sugahara[3]). *Let M be a complete noncompact Riemannian manifold of dimension $n \geqq 3$. Suppose that M satisfies the following three conditions.*
(i) *M has a pole o, i.e., the exponential mapping* $\exp_0$ *from the tangent space T_0M to M is a diffeomorphism.*
(ii) *$K_M \leqq k$ or $K_M \geqq k$ everywhere.*
(iii) $\liminf_{r \to \infty} \max_{d(o,p)=r} k(r)|K_M - k| = 0$, *where $d(o,p)$ denotes the distance between two points o and p and $k(r) = r^2$ for $k=0$ and $k(r) = \exp(2r\sqrt{-k})$ for $k<0$.*
Then M is isometric to the n-dimensional simply connected space of constant curvature k.

In this theorem, the third condition can be understood that M is in a neighbourhood of the model in the set of Riemannian manifolds. This theorem asserts that there are gaps in the positive and negative sides of the model spaces in the set of Riemannian manifolds. But the first condition implies that M is diffeomorphic to $\boldsymbol{R}^n$ and the assertion is restricted within metrics on $\boldsymbol{R}^n$.

We note that Gromov claimed that for M with $K_M \geqq k = 0$ the first condition can be relaxed to
(i′) *M is simply connected at infinity,*
where M is said to be simply connected at infinity if for any compact