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Structure and commutativity of rings with constraints involving a commutative subset

Dedicated to Professor Tosiro Tsuzuku on his 60th birthday

Hiroaki KOMATSU, Hisao TOMINAGA and Adil YAQUB

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Throughout, R will represent a ring with center C, N the set of nilpotent elements in R, N^* the subset of N consisting of all x with $x^2=0$. Given a positive integer n, we set $E_n=\{x\in R \mid x^n=x\}$; in particular, $E=E_2$. For $x, y\in R$, define extended commutators $[x, y]_k$ as follows: let $[x, y]_1$ be the usual commutator $[x, y]=xy-yx$, and proceed inductively $[x, y]_k=[[x, y]_{k-1}, y]$.

A ring R is called *nearly commutative* if R has no factorsubrings isomorphic to $M_\sigma(K)=\{\begin{pmatrix}\alpha & \beta\\ 0 & \sigma(\alpha)\end{pmatrix} \mid \alpha, \beta\in K\}$, where K is a finite field and σ is a non-trivial automorphism of K. Needless to say, every commutative ring is nearly commutative; every subring and every homomorphic image of a nearly commutative ring are nearly commutative. Following [2], R is called *s-unital* if for each x in R, $x\in Rx\cap xR$. As stated in [2], if R is an s-unital ring then for any finite subset F of R there exists an element e in R such that $ex=xe=x$ for all $x\in F$. Such an element e will be called a *pseudo-identity* of F.

Now, let A be a non-empty subset of R, and l a positive integer. We consider the following conditions:

(I$'$-A) For each $x\in R$, either $x\in C$ or there exists a polynomial $f(t)$ in $\boldsymbol{Z}[t]$ such that $x-x^2f(x)\in A$.

(II$'$-A) If $x, y\in R$ and $x-y\in A$, then either $x^m=y^m$ with some positive integer m or both x and y belong to the centralizer $C_R(A)$ of A in R.

(II-A)$_l$ If $x, y\in R$ and $x-y\in A$, then either $x^l=y^l$ or x and y both belong to $C_R(A)$.

(ii-A)$'$ For each $x\in R$ and $a\in A$, there exists a positive integer m, depending on x and a, such that $[a, x^m]=0$.

(ii-A)$'_l$ $[a, x^l]=0$ for all $x\in R$ and $a\in A$.

(ii-A)* For each $x\in R$ and $a\in A$, there exist positive integers k and m, each depending on x and a, such that $[a, x^m]_k=0$.