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The Grothendieck ring of linear representations of a finite category

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Dedicated to Professor Tosiro Tsuzuku
on his sixtieth birthday

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Introduction

A finite category is a category whose objects and morphisms form finite sets. Yoshida proved the following theorem in his attempt to define the Burnside ring of a finite category [4].

THEOREM. *Suppose that a finite category C satisfies the following conditions.*

(a) *C has the unique epi-mono factorization property (see Section 4 for the precise definition).*

(b) *For any object x of C and any cyclic subgroup H of* $\mathrm{Aut}(x)$, *a quotient object $H\backslash x$ exists.*

Let I be a set of representatives for isomorphism classes of objects of C. Denote by $\mathbf{Z}[I]$ and $\mathbf{Z}^I$ the free abelian group on I and the ring of $\mathbf{Z}$-valued functions on I respectively. Define a group homomorphism φ: $\mathbf{Z}[I]\longrightarrow\mathbf{Z}^I$ by $\varphi(x)(y)=\#\mathrm{Hom}_C(y, x)$ for $x, y\in I$. Then

(i) *φ is injective.*

(ii) $\#\mathrm{Coker}(\varphi)=\prod_{x\in I}\#\mathrm{Aut}(x)$.

(iii) $\mathrm{Image}(\varphi)$ *is a subring of $\mathbf{Z}^I$ (with the common identity).*

Thus, for such a category C, $\mathbf{Z}[I]$ has a unique ring structure such that φ is a ring homomorphism. Yoshida called $\mathbf{Z}[I]$ the abstract Burnside ring of C. When C is the category of transitive G-sets for a finite group G, the ring $\mathbf{Z}[I]$ is just the Burnside ring of G, i.e., the Grothendieck ring of the category of finite G-sets.

In this paper we prove a linear version of the above theorem. Let k be a field of characteristic $p>0$ and C a finite category. A $k[C]$-module means a functor $C^{\mathrm{op}}\longrightarrow\{k\text{-modules}\}$. Let $G_0(k[C])$ (resp. $K_0(k[C])$) be the Grothendieck group of the category of finite dimensional (resp. finite dimensional projective) $k[C]$-modules with respect to exact sequences. Tensor product makes $G_0(k[C])$ a commutative ring. Let $c: K_0(k[C])$