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Two notes on conformal geometry

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Dedicated to Professor Noboru Tanaka
on the occassion of his sixtieth birthday

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This paper provides new approaches to two old results in the study of conformal mappings of Euclidean space.

Part 1. Liouville's theorem on conformal maps

Liouville's theorem states that a conformal map between open sets in a Euclidean space E^n, $n\geq 3$ may be extended, after allowing that the map might take on the point at infinity as a value, to a global conformal map; and that the group of such conformal maps is finite dimensional. We show how to simplify the standard proofs of this theorem using an elementary but apparently new observation that the normal curvature of a curve in a hypersurface changes in a very transparent way under conformal mappings.

There are two common ways of stating Liouville's result. For the first, recall that the inversions, dilations, translations, and orthogonal rotations on $E^n\cup\{\infty\}$ generate a finite dimensional group called the Möbius group, $M(n)$.

THEOREM 1. *For $n\geq 3$, any conformal map of open sets in E^n is the restriction of an element of $M(n)$.*

For the second statement, we identify the conformal structure on E^n with that of the sphere (with one point deleted). We then identify S^n with the standard hyperquadric of homogeneous signature $(n+1, 1)$ in the real projective space P^{n+1} of one higher dimension and produce an action of $Q(n+1, 1)$ on the sphere and thus on the extended Euclidean plane.

THEOREM 2. *For $n\geq 3$, any conformal map of open sets in E^n is the restriction of an element of $O(n+1, 1)$. For $n\geq 3$ and n odd, the group of conformal maps is isomorphic to $SO(n+1, 1)$. For $n\geq 3$ and n even, the group of conformal maps is isomorphic to $(O(n+1, 1)/Z_2)\ltimes Z_2$ for a*

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