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On a division ring with discrete valuation

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Throughout this paper A will be a division ring with non-trivial valuation v, and C will be the center of A. A has the completion with respect to the v-topology, which is also a division ring. We will denote it by A^*. For each division subring B of A the closure of B in A^* with respect to the v-topology will be also denoted by B^*. B^* is isomorphic to the completion of B as topological ring.

The aim of this paper is to show that B^* coincides with the double centralizer of B in A^* for each division C-subalgebra B of A, in the case where A is finite over C and v is discrete.

In this paper we will use the same terminology as [4] and [6]. In particular for each division subring B of A we write

$$O(B)=\{x\in B|v(x)\leqq 1\},\ P(B)=\{x\in B|v(x)<1\}.$$

Let v be non-archimedean and B an arbitrary division subring of A. $O(B)$ is a local ring with the maximal ideal $P(B)$. Hence $O(B)/P(B)$ is a division ring, which will be denoted by $E(B)$. Write $E(B)=K$ and $E(A)=E$. Then K is a division subring of E. We will write $f_r(A/B)=[E:K]_r$, $f_l(A/B)=[E:K]_l$ and $e(A/B)=[v(A^\circ):v(B^\circ)]$, where A° and B° are the unit groups of A and B, respectively. In the case where $[E:K]_l=[E:K]_r$, we will write $f(A/B)$ in stead of $f_r(A/B)$ or $f_l(A/B)$. Note that we have $e(A^*/A)=f(A^*/A)=1$ by Proposition 17.4 and Corollary 17.4 b [4]. Furthermore the Domination Principle, that is, $v(x)<v(y)$ implies $v(x+y)=v(y)$, holds also for a division ring with non-Archimedean valuation (See § 17.2 [4]).

The next lemma is well known in the case where A is a commutative field, and holds also in the case where $v|B$ is trivial

LEMMA 1. *Let A, B and v be as above, then we have $e(A/B)f_r(A/B)\leqq[A:B]_r$. If $[A:B]_r<\infty$, both $e(A/B)$ and $f_r(A/B)$ are finite.*

PROOF. Since $v(ab)=v(a)v(b)=v(b)v(a)=v(ba)$ for any a, $b\in A$, and the Domination Principle holds for A, we can follow the same lines as the proof of Theorem 4.5 Chap. 2 [2].